What is resonance?

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Resonance

Linear or nonlinear?
That is the question...

Linear resonance

Consider a periodically forced harmonic oscillator

$$mx'' = -\lambda x + e(t)\,.$$

Here $m > 0$ is the mass, $\lambda > 0$ is the spring constant, and

$e(t)$ is a T-periodic external force.

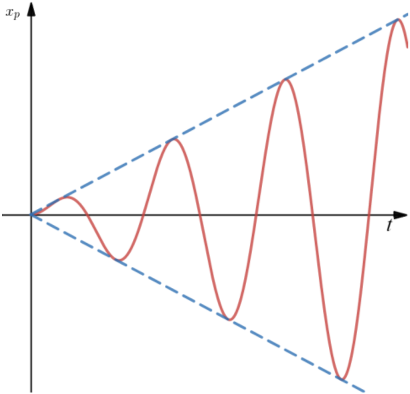
For simplicity, let $m = 1$ and $T = 2\pi$.

We can write the Fourier expansion

$$e(t) \sim \sum_{n=0}^{\infty} \Big(a_n \cos(nt) + b_n \sin(nt) \Big)\,.$$

If $\lambda = n^2$ for some $n \in \mathbb{N}$, and either $a_n \neq 0$ or $b_n \neq 0$, then *all solutions are unbounded*.

Linear resonance



This is what we call “resonance”.

Nonlinear resonance

When the linear term λx is replaced by a nonlinear one, the situation is much more complicated. Consider for example an equation like

$$mx'' = -g(x) + e(t)\,,$$

where

$$g(x) = \begin{cases} \mu x\,, & \text{if } x \geq 0\,, \\ \nu x\,, & \text{if } x < 0\,, \end{cases}$$

with $\mu > 0$, $\nu > 0$, and

$$e(t) \text{ is a } T\text{-periodic external force.}$$

It has been proved that, in some cases,

<u>there exists a periodic solution</u> while
<u>all large amplitude solutions are unbounded</u>

(Fabry–F. and Alonso–Ortega, 1998)

Nonlinear resonance

Nonlinear resonance is ubiquitous in nature.

It is relevant to understanding phenomena in quite distinct areas such as music, mechanics, engineering, astronomy, ...

Many sounds we hear, such as our voice, or when objects of metal, glass, or wood are struck, are caused by resonant vibrations.

Light and other electromagnetic radiation is produced by resonance on an atomic scale, such as electrons in atoms.

Optical resonance is used to create coherent light in a laser cavity.

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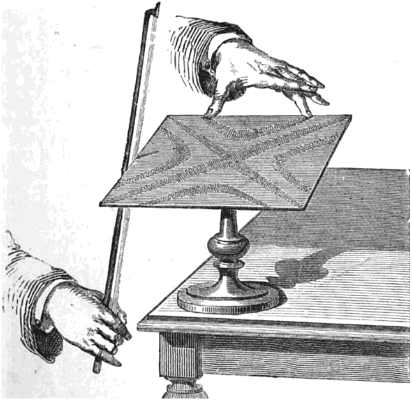
Exploiting resonance

Exploiting resonance

Musical instruments.

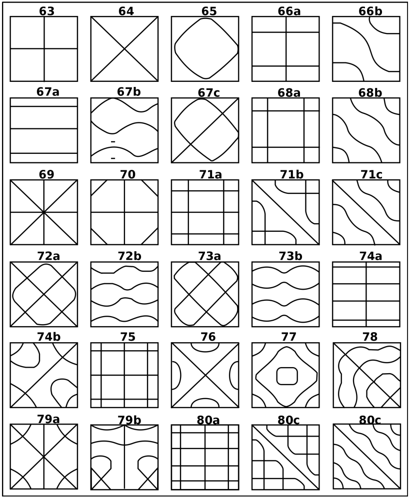
Acoustic resonances of musical instruments can produce pleasant sounds, melodies, sonatas, great symphonies...

Chladni's patterns



Ernst Chladni, 1787.

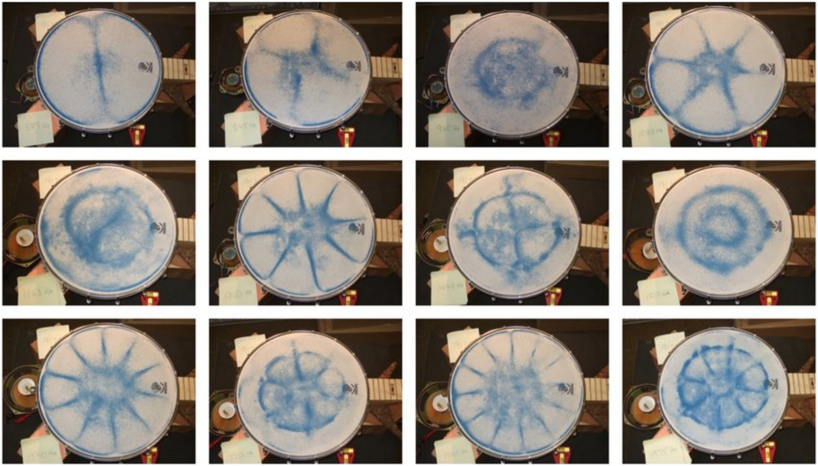
Chladni's patterns



Ernst Chladni, 1787.

Chladni's patterns in violins

Chladni's patterns in drums



A good question

Can one hear the shape of a drum?

"Can one hear the shape of a drum?" is the title of an article by Mark Kac published in 1966 in the American Mathematical Monthly.

The frequencies at which a drumhead can vibrate depend on its shape. These frequencies are the eigenvalues of the Laplace operator with Dirichlet boundary conditions

$$\begin{cases} -\Delta u = \lambda u \\ u|_{\partial D} = 0 \end{cases}$$

The question is whether the shape can be predicted if the frequencies are known.

In 1988 Catherine Durso proved (in her PhD thesis) that the question has a positive answer for triangles.

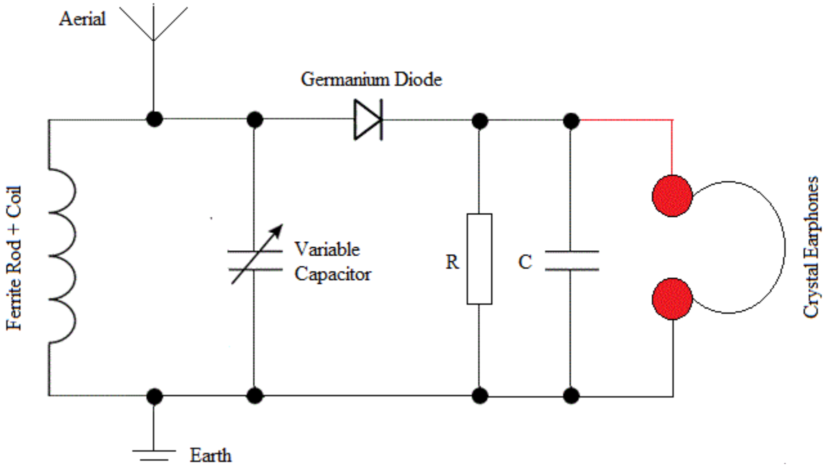
The general problem in two dimensions remained open until 1992, when Carolyn Gordon, David Webb, and Scott Wolpert constructed a pair of regions in the plane that have different shapes but identical eigenvalues. The regions are concave polygons.

On the other hand, Steve Zelditch in 2000 proved that the answer to Kac's question is positive if one imposes restrictions to certain convex planar regions with analytic boundary.

It is not known whether two non-convex analytic domains can have the same eigenvalues.

Exploiting resonance

Electrical resonance of tuned circuits in radios and televisions.



Exploiting resonance

Nuclear magnetic resonance.

Exploiting resonance

Timekeeping mechanisms of clocks and watches.

(both mechanical watches and quartz watches)

Tidal resonance

Bay of Fundy (Canada)

Resonance can be destructive!

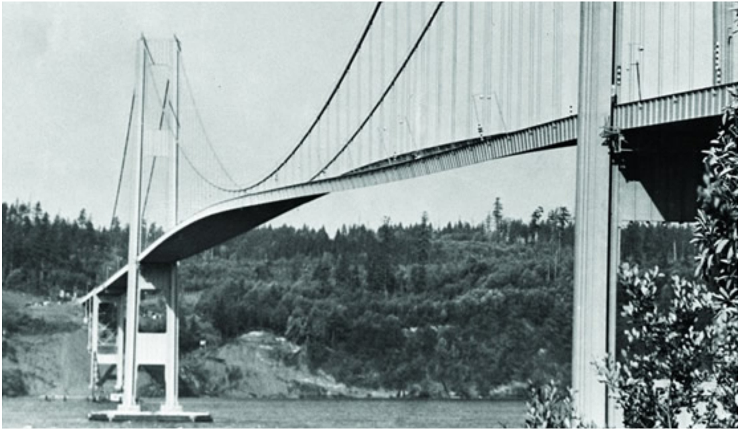
Resonance can be destructive

Shattering of a crystal wineglass.

(when exposed to a musical tone of its resonant frequency)

Resonance in suspension bridges

Tacoma Narrows bridge, 1940.



Resonance in suspension bridges

After 82 years, there is no unanimously accepted explanation of the reasons of the Tacoma Narrows bridge collapse.
And many further bridges failures still remain without explanation.

Resonance in suspension bridges

A simplified model has been proposed, of the type

$$mx'' = -g(x) + e(t)\,,$$

where

$$g(x) = \begin{cases} \mu x\,, & \text{if } x \geq -h\,, \\ \nu(x+h) - \mu h\,, & \text{if } x < -h\,, \end{cases}$$

and

$e(t)$ is a T-periodic external force.

The existence of large amplitude *subharmonic solutions* has been proved, when $\nu > 0$ is sufficiently small.

(F.–Ramos and F.–Schneider–Zanolin, 1994)

Resonance in racing cars and motorboats

The porpoising effect.

Porpoising is a bouncing effect classically observed in motorboats that above a critical speed start an oscillatory motion leaping out and striking the water alternatively.

More recently, this term has become familiar among the Formula 1 racing fans.

During the first races of the 2022 season most of the cars were subject to rather violent periodic bounces.

It is clear that not only the car mechanics is compromised, but also the driver's safety.

Resonance in racing cars and motorboats

The porpoising effect.

A model has been proposed, of the type

$$mx'' = -g(x) + e(t)\,,$$

where

$$e(t) \text{ is a } T\text{-periodic external force,}$$

but now the function $g(x)$ is discontinuous at $x = 0$.

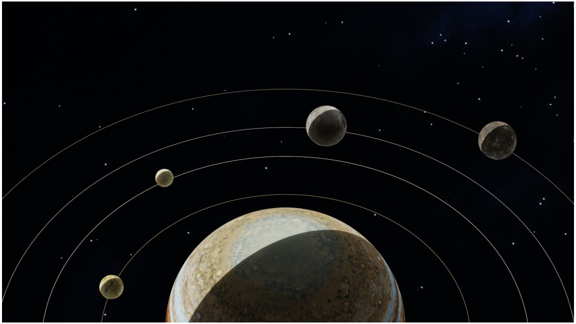
Still, the existence of large amplitude *subharmonic solutions* has been proved.

(F.–Torres, preprint 2022)

Celestial Mechanics

Resonances in the solar system

The Jupiter's moons.



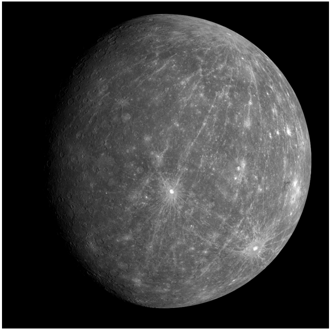
The most massive ones are the four Galilean moons:

Io, Europa, Ganymede, and Callisto,

(from smaller to larger distance from Jupiter). The periods of revolution of the first three are in exact $1:2:4$ resonance!

Resonances in the solar system

Mercury's rotation.



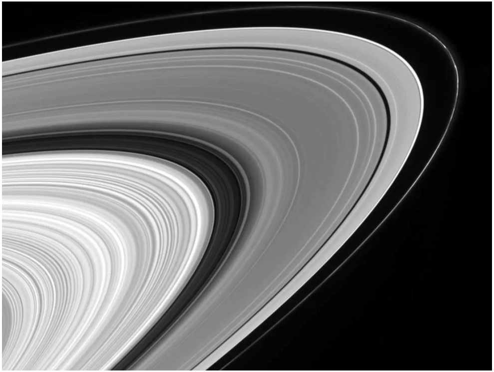
The period of revolution around the sun is exactly $3 : 2$ the period of rotation around its axis (in the same direction of rotation).This is an

orbital-rotational resonance!

Resonances in the solar system

Saturn's rings.

Resonances with the planet's moons create gaps in the rings.



Many of these gaps however have found no explanation till now.

The periodically perturbed Kepler problem

The classical Kepler problem is modeled by Newton's equation

$$x'' = -G\,m\,\frac{x}{|x|^3}\,.$$

The solutions are well known (ellipses, parabolas or hyperbolas).

If we perturb the gravitational force by a small T-periodic term

$$x'' = -(1 + \varepsilon e(t))\,G\,m\,\frac{x}{|x|^3}\,,$$

the existence of periodic solutions can be proved.

(F.–Toader 2008, 2012; F.–Ureña 2011; F.–Gallo 2017, 2018)

The N-body problem

The problem is modeled by the system

$$x_i'' = -G \sum_{\substack{k=1 \\ k\neq i}}^{N} m_k \, \frac{x_i - x_k}{|x_i - x_k|^3}\,, \quad i = 1, \ldots, N\,.$$

When $N \geq 3$, very little is known. There could be

periodic configurations, but also chaotic dynamics.

This can lead to collisions, or even

solutions which escape to infinity in finite time!

Paul Painlevé proved in 1897 that this never happens if $N = 3$, and he raised the conjecture for $N \geq 4$. The conjecture was proved by Jeff Xia for $N \geq 5$ in 1988, and by Jinxin Xue for $N = 4$ in 2014.

Will resonance destroy the solar system?

This is one of the main open problems in Celestial Mechanics.

Thank you for listening!

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