Existence and Non-Existence of Solutions of Third Order Equations Coupled to Three-Point Boundary Conditions

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Abstract

In this talk, we present existence and non existence results for the third order nonlinear differential equation

$$u'''(t) = -\lambda\, p(t)\, f(u(t)), \text{ a.e. } t \in I := [0, 1], \tag{1}$$

coupled to the three-point boundary value conditions

$$u(0) = 0, \; u''(\eta) = \alpha\, u'(1), \; u'(1) = \beta\, u(1), \tag{2}$$

with $0 \leq \alpha \leq 1, 0 \leq \beta < \frac{2}{2-\alpha}$ and $0 \leq \eta \leq \frac{1}{3}$.

Abstract

Taking into account that the related Green's function is nonpositive for $0 \leq s < \eta$ and nonnegative if $\eta < s \leq 1$, we assume the following conditions on the nonlinear part of the equation:

(F) $\lambda > 0$ is a parameter, $p \in L^\infty(I)$ is such that $p < 0$ a.e. on $[0, \eta]$ and $p > 0$ a. e. on $[\eta, 1]$ and $f : [0, \infty) \to [0, \infty)$ is a continuous function.

ABSTRACT

By defining suitable cones on $C^1(I)$, under additional conditions on the asymptotic behavior of function f, we deduce, for a particular set of values of the positive parameter λ, the existence of positive and increasing solutions on the whole interval of definition which are convex on $[0, \eta]$. The results hold by means of degree theory.

A. C., N. D. Dimitrov, *Third-order differential equations with three-point boundary conditions.* Open Math. **19** (2021), 1, 11–31.

PARTS OF THE TALK

- INTRODUCTION

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- LINEAR PROBLEM

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- Introduction
- Linear Problem
- Nonlinear Problem

Part I

INTRODUCTION

Third order three-point boundary value problems arise in several areas of applied mathematics and physics: some particular models of deflection of a curved beam with a constant or varying cross sections, three layer beams, electromagnetic waves, study of the equilibrium states of a hated bar and others.

M. Greguš, *Third Order Linear Differential Equations*, Mathematics and its Applications, D. Reidel Publishing Co., Dordrecht, 1987.

Using Krasnosels'kii's fixed-point theorem, Sun proved the existence of infinite positive solutions of the BVP

$$\begin{aligned} u'''(t) &= \lambda\, a(t)\, f(t, u(t)),\ 0 < t < 1, \\ u(0) &= u'(\eta) = u''(1) = 0,\ \eta \in (1/2, 1), \end{aligned}$$

assuming that f is sublinear or superlinear with respect to the second variable.

Y. Sun, *Positive solutions of singular third-order three-point boundary-value problem*, J. Math. Anal. Appl. **306** (2005), 589-603.

Liu et al. studied the above problem with two-point boundary conditions

$$u(0) = u(1) = u''(1) = 0 \text{ and } u(0) = u'(1) = u''(0) = 0.$$

Z. Q. Liu, J. S. Ume, and S. M. Kang, *Positive solutions of a singular nonlinear third-order two-point boundary value problem*, J. Math. Anal. Appl. **326** (2007), 589-601.

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Z. Q. Liu, J. S. Ume, D. R. Anderson, and S. M. Kang, *Twin monotone positive solutions to a singular nonlinear third-order differential equation*, J. Math. Anal. Appl. **334** (2007), 299-313.

The three-point boundary value problem

$$u(0) = u'(0) = 0, \; u'(1) = \alpha\, u'(\eta), \; 0 < \eta < 1, \; 1 < \alpha < 1/\eta,$$

is considered in

L. J. Guo, J. P. Sun, and Y. H. Zhao, *Existence of positive solutions for nonlinear third-order three-point boundary value problem*, Nonlinear Anal. **68** (2008), 3151-3158.

In all the previous mentioned papers, the existence of positive solution follows from the fact that the corresponding Green's function is strictly positive.

Palamides and Veloni studied the singular BVP

$$\begin{aligned} u'''(t) &= -a(t)\, f(t, u(t)),\ 0 < t < 1, \\ u(0) &= u'(1) = u''(\eta) = 0,\ \eta \in [0, 1/2]. \end{aligned}$$

The corresponding Green's function G has not constant sign.

However, the solution

$$u(t) = \int_0^1 G(t,s)\, a(s)\, f(s, u(s))\, ds$$

may be positive if its initial values $u'(0)$ and $u''(0)$ are positive.

A. P. Palamides, A. N. Veloni, *A singular third-order boundary-value problem with nonpositive Green's function*, Electron. J. Differential Equations **2007** (2007), no. 151, 1-13.

Part II

LINEAR PROBLEM

Consider, for any $y \in C(I)$, the following three-point linear boundary value problem

$$
\begin{aligned}
-u'''(t) &= y(t), \; 0 \le t \le 1, && (3)\\
u(0) &= 0, \; u''(\eta) = \alpha\, u'(1), \; u'(1) = \beta\, u(1), && (4)
\end{aligned}
$$

with $0 \le \alpha \le 1, 0 \le \beta < \frac{2}{2-\alpha}$ and $0 \le \eta \le \frac{1}{2}$.

Consider, for any $y \in C(I)$, the following three-point linear boundary value problem

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with $0 \le \alpha \le 1, 0 \le \beta < \frac{2}{2-\alpha}$ and $0 \le \eta \le \frac{1}{2}$.

It is immediate to verify that this problem has a unique solution if and only if

$$\beta\,(2-\alpha) \neq 2.$$

A. C., L. López-Somoza, M. Yousfi, *Green's Function Related to a n-th Order Linear Differential Equation Coupled to Arbitrary Linear Non-Local Boundary Conditions*, Mathematics 2021, 9(16), 1948.

$$\begin{cases} L_n u(t) = y(t), & t \in I, \\ B_i(u) = \delta_i C_i(u), & i = 1, \dots, n, \end{cases} \tag{5}$$

$$L_n u(t) := u^{(n)}(t) + a_1(t)\, u^{(n-1)}(t) + \cdots + a_n(t)\, u(t), \quad t \in I.$$

Here y and a_k are continuous functions for all $k = 0, \dots, n-1$ and $\delta_i \in \mathbb{R}$ for all $i = 1, \dots, n$.

$C_i : C^n(I) \to \mathbb{R}$ is a linear continuous operator and B_i covers the general two point linear boundary conditions, i.e.:

$$B_i(u) = \sum_{j=0}^{n-1} \left(\alpha_j^i u^{(j)}(0) + \beta_j^i u^{(j)}(1) \right), \quad i = 1, \dots, n,$$

being α_j^i, β_j^i real constants for all $i = 1, \dots, n$, $j = 0, \dots, n-1$.

LEMMA

There exists the unique Green's function g related to

$$\begin{cases} L_n u(t) = y(t), & t \in I, \\ B_i(u) = 0, & i = 1, \ldots, n, \end{cases} \tag{6}$$

if and only if for any $i \in \{1, \cdots, n\}$, the following problem

$$\begin{cases} L_n u(t) = 0, & t \in I, \\ B_j(u) = 0, & j \neq i, \\ B_i(u) = 1, & \end{cases}$$

has a unique solution, that we denote as $\omega_i(t)$, $t \in I$.

THEOREM

Assume that Problem (6) *has a unique solution and let* g *be its related Green's function. Let* δ_i, $i = 1, \ldots, n$, *be such that*

$$\det(I_n - A) \neq 0,$$

with I_n *the identity matrix and* $A = (a_{ij})_{n\times n} \in \mathcal{M}_{n\times n}$ *given by*

$$a_{ij} = \delta_j\, C_i(\omega_j), \quad i,\ j \in \{1, \ldots, n\}.$$

Then Problem (5) *has a unique solution with Green's function*

$$G\,(t, s, \delta_1, \ldots, \delta_n) := g\,(t, s) + \sum_{i=1}^{n}\sum_{j=1}^{n} \delta_i\, b_{ij}\, \omega_i(t)\, C_j\,(g(\cdot, s))\,, \quad t,\ s \in I,$$

with ω_j *defined on Lemma 1 and* $B = \left(b_{ij}\right)_{n\times n} = (I_n - A)^{-1}$.

In our particular case, we can rewrite Problem (3)–(4), as

$$\begin{cases} L_n u\,(t) = u'''(t) = -y(t), \quad t \in I, \\ B_1\,(u) = u(0) = \delta_1\, C_1\,(u) = 0, \\ B_2\,(u) = u'(1) = \delta_2\, C_2\,(u) = \frac{1}{\alpha}\, u''(\eta), \\ B_3\,(u) = u(1) = \delta_3\, C_3\,(u) = \frac{1}{\beta}\, u'(1). \end{cases}$$

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$$g(t,s) = \begin{cases} -\frac{1}{2} s^2 (t-1)^2, & 0 \le s \le t \le 1, \\ -\frac{1}{2}(s-1)t(s(t-2)+t), & 0 < t < s \le 1. \end{cases}$$

$$w_1(t) = t^2 - 2t + 1, \quad w_2(t) = t^2 - t, \quad w_3(t) = 2t - t^2.$$

$$\det(I_n - A) = \frac{(\alpha-2)\beta + 2}{\alpha\beta} \; (\neq 0).$$

$$
\begin{aligned}
C_1(g(\cdot,s)) &= 0 \\
C_2(g(\cdot,s)) &= \frac{\partial^2 g}{\partial t^2}(\eta,s) = \begin{cases} -s^2 & s<\eta, \\ 1-s^2, & \eta<s. \end{cases} \\
C_3(g(\cdot,s)) &= \frac{\partial g}{\partial t}(1,s) = 0
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\end{aligned}
$$

$$
\begin{aligned}
G(t,s) &= g(t,s) + (b_{22}\,w_2(t) + b_{32}\,w_3(t))\;C_2(g(\cdot,s))\\
&= \begin{cases} -\frac{1}{2}s^2(t-1)^2, & 0\le s\le t\le 1,\\ -\frac{1}{2}(s-1)t(s(t-2)+t), & 0<t<s\le 1.\end{cases}\\
&\quad + \frac{t(\beta(t-1)+t-2)}{(\alpha-2)\beta+2}\begin{cases} -s^2 & s<\eta,\\ 1-s^2, & \eta<s.\end{cases}
\end{aligned}
$$

If $s > \eta$, then

$$G(t,s) = \begin{cases} \frac{\alpha\beta\left(1-s^2\right)}{2(2+\alpha\beta-2\beta)}t^2 + \left(\frac{2-\beta}{2+\alpha\beta-2\beta} - s - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta}s^2\right)t, & s > t, \\ \frac{2\beta-2-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)}t^2 + \left(\frac{2-\beta}{2+\alpha\beta-2\beta} - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta}s^2\right)t - \frac{s^2}{2}, & s \leq t. \end{cases}$$

If $s < \eta$, then

$$G(t,s) = \begin{cases} \frac{2+\alpha\beta-2\beta-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)}t^2 - \left(s + \frac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta}\right)t, & s > t, \\ \frac{-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)}t^2 - \frac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta}t - \frac{s^2}{2}, & s \leq t. \end{cases}$$

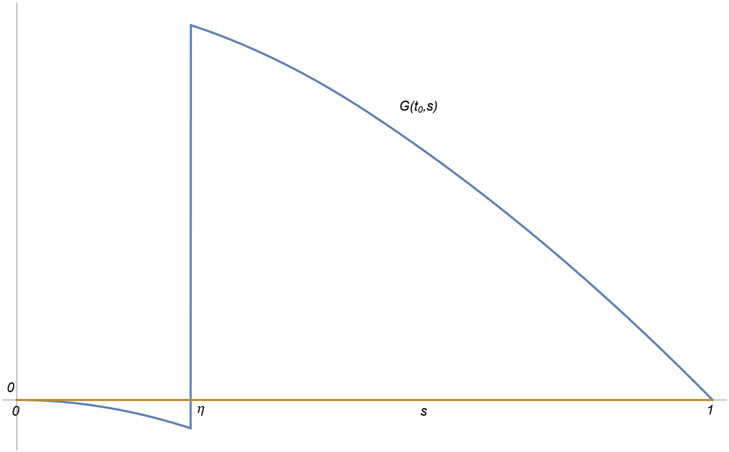

FIGURE: Graph of $G(t_0, s)$, $s \in [0, 1]$, with $t_0 \in (0, 1)$ fixed.

Remark

We point out that on our calculations we have assumed that $\alpha \neq 0$, $\beta \neq 0$ *and* $2 + \alpha\beta - 2\beta \neq 0$. *Moreover the expression is valid for* $\alpha = 0$ *or* $\beta = 0$. *In particular, if* $\alpha = \beta = 0$, *we obtain the expression of the Green's function given in*

REMARK

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A. P. Palamides, A. N. Veloni, A singular third-order boundary-value problem with nonpositive Green's function, Electron. J. Differential Equations **2007** *(2007), no. 151, 1-13.*

LEMMA

Let $0 \leq \alpha \leq 1$, $0 \leq \beta < \frac{2}{2-\alpha}$ and $0 \leq \eta \leq \frac{1}{2}$. The Green's function G, related to problem $(3) - (4)$, has the following sign properties:

$$G(t,s) \leq 0 \text{ and } \frac{\partial}{\partial t} G(t,s) \leq 0 \qquad \text{for } 0 \leq s < \eta,$$
$$G(t,s) \geq 0 \text{ and } \frac{\partial}{\partial t} G(t,s) \geq 0 \qquad \text{for } \eta < s \leq 1.$$

$$\frac{\partial^2}{\partial t^2} G(t,s) \leq 0 \qquad \text{for all } s < t,$$
$$\frac{\partial^2}{\partial t^2} G(t,s) \geq 0 \qquad \text{for all } s > t.$$

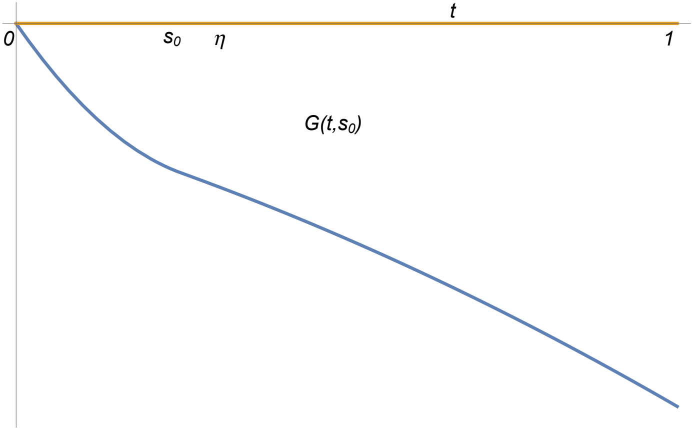

Figure: Graph of $G(t, s_0)$, $0 < s_0 < \eta < 1$ fixed.

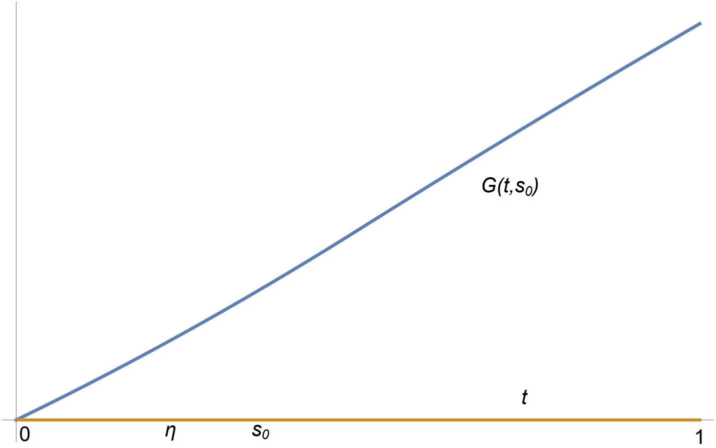

FIGURE: Graph of $G(t, s_0)$, $0 < \eta < s_0 < 1$ fixed.

LEMMA

$$\max_{t,s\in I}\left|\frac{\partial}{\partial t}G(t,s)\right| \leq \frac{2+\alpha\beta-\beta}{2+\alpha\beta-2\beta}.$$

Now, we define the cone

$$K := \left\{ y \in C^1(I) : \; y(t) \geq 0, \; y'(t) \geq 0, t \in I \right\}.$$

LEMMA

Let $0 \leq \alpha \leq 1, 0 \leq \beta < \frac{2}{2-\alpha}, 0 \leq \eta \leq \frac{1}{2}$ and G be the related Green's function to Problem (3) – (4). Let $y \in K$. Then the unique solution of the linear boundary value Problem (3)–(4) is such that

$$u(t) = \int_0^1 G(t,s)\, y(s)\, ds \in K.$$

Moreover, $u \in C^2(I)$ and $u''(t) \geq 0$ for all $t \in [0, \eta]$.

Idea of the Proof

We only show how to deduce that $u(t) \geq 0$ for all $t \in [0, \eta]$.

To this end we use that $G(t,s) \leq 0$ for $0 \leq s < \eta$ and $G(t,s) \geq 0$ for $\eta < s \leq 1$, $y(t) \geq 0$, $y'(t) \geq 0$ for $0 \leq t \leq 1$, we have

$$\begin{aligned} u(t) &= \int_0^\eta G(t,s)\, y(s)\, ds + \int_\eta^1 G(t,s)\, y(s)\, ds \\ &\geq \max_{0 \leq s \leq \eta} y(s) \int_0^\eta G(t,s) ds + \min_{\eta \leq s \leq 1} y(s) \int_\eta^1 G(t,s) ds \\ &= y(\eta) \int_0^\eta G(t,s) ds + y(\eta) \int_\eta^1 G(t,s) ds \\ &\geq y(\eta) \left(-\frac{1}{6} t \frac{-6 + \alpha\beta - 6\beta\eta + 6\eta\beta t + 2\beta - 2\beta t \alpha - 6t\eta + 12\eta + \alpha\beta t^2 + 2t^2 - 2\beta t^2}{2 + \alpha\beta - 2\beta} \right) \\ &\geq 0. \end{aligned}$$

Idea of the Proof

The rest of the properties on u follow with similar arguments, by using that $y \in K$ and the sign properties of G and its two first partial derivatives.

Now, define $h(t) := 1 + \alpha\,(t-1)$. So we obtain

LEMMA

Let $0 \le \alpha \le 1, 0 \le \beta < \frac{2}{2-\alpha}$ *and* $0 \le \eta \le \frac{1}{2}$. *Then*

$$\frac{\partial}{\partial t} G(t,s) \le h(t) \frac{\partial}{\partial t} G(1,s) \qquad \text{for } 0 \le s < \eta,$$

$$\frac{\partial}{\partial t} G(t,s) \ge h(t) \frac{\partial}{\partial t} G(1,s) \qquad \text{for } \eta < s \le 1.$$

Moreover, given $y \in K$, *the unique solution u of Problem* (3)–(4), *is such that*

$$u'(t) \ge h(t)\, u'(1) \text{ for all } t \in I.$$

LEMMA

Let $0 \leq \alpha \leq 1, 0 \leq \beta < \frac{2}{2-\alpha}$ *and* $0 \leq \eta < \frac{1}{2}$ *and G be the related Green's function to Problem* (3) – (4). *Then, for all* $(t,s) \in (0,1] \times (0,1)$ *the following inequalities are fulfilled:*

$$\frac{G(t,s)}{G(1,s)} \leq \lim_{s \to 0^+} \frac{G(t,s)}{G(1,s)} \leq \frac{1}{2}\beta(t-1)(\alpha(t-1)+2)+1 \leq 1$$

and

$$\frac{G(t,s)}{G(1,s)} \geq \lim_{s \to 1^-} \frac{G(t,s)}{G(1,s)} = \frac{1}{2}\alpha\beta(t-1)t+t =: g(t).$$

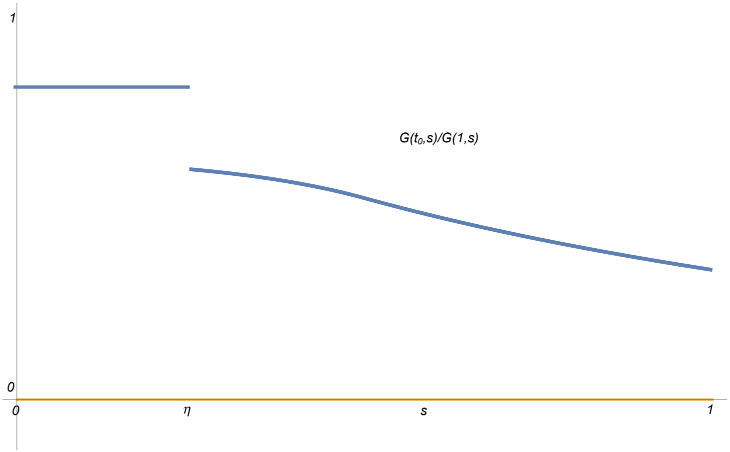

FIGURE: Graph of $G(t_0, s)/G(1, s)$, $s \in [0, 1]$, with $t_0 \in (0, 1)$ fixed.

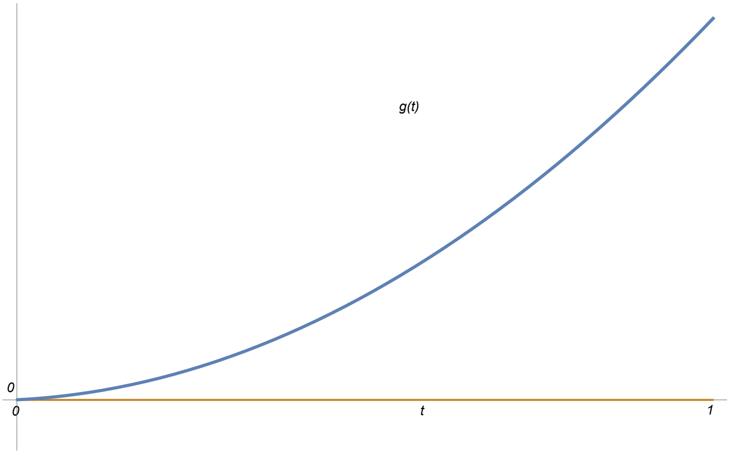

FIGURE: Graph of function g.

COROLLARY

Let $0 \leq \alpha \leq 1, 0 \leq \beta < \frac{2}{2-\alpha}$ *and* $0 \leq \eta \leq \frac{1}{2}$. *Then*

$$G(t,s) \leq g(t)\, G(1,s) \qquad \text{for } 0 \leq s < \eta,$$

$$G(t,s) \geq g(t)\, G(1,s) \qquad \text{for } \eta < s \leq 1.$$

$$K_0 := \{y \in K,\ y(t) \geq g(t)\, \|y\|_\infty,\ t \in I\}.$$

So, we deduce that the solution of (3)– (4) belongs to the previous cone, when η is in the more restrictive interval $[0, 1/3]$.

Lemma

Let $0 \leq \alpha \leq 1, 0 \leq \beta < \frac{2}{2-\alpha}$, $0 \leq \eta \leq \frac{1}{3}$ *and G be the related Green's function to problem* (3) – (4)*. Let* $y \in K_0$*. Then the unique solution of the linear boundary value problem* (3)– (4) *is such that*

$$u(t) = \int_0^1 G(t,s)\, y(s)\, ds \in K_0.$$

Part III

Non Linear Problem

Now we will study the existence of solutions of the third order nonlinear differential equation

$$u'''(t) = -\lambda\, p(t)\, f(u(t)), \text{ a.e. } t \in I, \tag{1}$$

$$u(0) = 0,\ u''(\eta) = \alpha\, u'(1),\ u'(1) = \beta\, u(1), \tag{2}$$

with $0 \le \alpha \le 1, 0 \le \beta < \frac{2}{2-\alpha}$ and $0 \le \eta \le \frac{1}{3}$.

(F) $\lambda > 0$ is a parameter, $p \in L^\infty(I)$ is such that $p < 0$ a.e. on $[0, \eta]$ and $p > 0$ a. e. on $[\eta, 1]$ and $f : [0, \infty) \to [0, \infty)$ is a continuous function.

Let us consider the Banach space $C^1(I)$ equipped with the norm

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}.$$

Taking into account the properties satisfied by the Green's function and its derivatives, we define the cone K_1 in $C^1(I)$ as follows

$$K_1 := \left\{y \in K_0,\ y'(t) \geq h(t)\, y'(1),\ t \in I\right\},$$

$$K_0 := \{y \in K,\ y(t) \geq g(t)\, \|y\|_\infty,\ t \in I\},$$

$$K := \left\{y \in C^1(I) :\ y(t) \geq 0,\ y'(t) \geq 0, t \in I\right\}.$$

It is well known that the solutions of Problem (1)-(2) correspond with the fixed points of the integral operator

$$Tu(t) = \lambda \int_0^1 G(t,s)\, p(s)\, f(u(s))\, ds, \;\; t \in I.$$

Lemma

$T : K_1 \to K_1$ is a completely continuous operator.

Define

$$\Lambda = \int_0^1 G(1,s)\, p(s)\, g(s) ds > 0,$$

$$p^* = \text{sup ess}_{\, s \in I} |p(s)|$$

and denote, assuming that both limits exist,

$$f_0 = \lim_{x \to 0^+} \frac{f(x)}{x} \text{ and } f^\infty = \lim_{x \to +\infty} \frac{f(x)}{x}.$$

Theorem (1)

Assume that $0 \leq \alpha \leq 1, 0 \leq \beta < \frac{2}{2-\alpha}, 0 \leq \eta \leq \frac{1}{3}$ *and*

$$\frac{2+\alpha\beta-\beta}{2+\alpha\beta-2\beta} f^{\infty} p^{*} < \Lambda f_0.$$

Then, if

$$\lambda \in \left(\frac{1}{\Lambda f_0}, \frac{2+\alpha\beta-2\beta}{(2+\alpha\beta-\beta) f^{\infty} p^{*}} \right)$$

Problem (1)-(2) *has at least one positive solution in* K_1.

Idea of the Proof

Assume, at first, that $f_0 \in (0, +\infty)$.

Let $\lambda \in \left(\frac{1}{\Lambda f_0}, \frac{2+\alpha\beta-2\beta}{(2+\alpha\beta-\beta) f^\infty p^*} \right)$ and choose $\varepsilon \in (0, f_0)$ such that

$$\frac{1}{\Lambda (f_0 - \varepsilon)} < \lambda < \frac{2 + \alpha\beta - 2\beta}{(2 + \alpha\beta - \beta)(f^\infty + \varepsilon) p^*}.$$

Idea of the Proof

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Let $\lambda \in \left(\frac{1}{\Lambda f_0}, \frac{2+\alpha\beta-2\beta}{(2+\alpha\beta-\beta) f^\infty p^*} \right)$ and choose $\varepsilon \in (0, f_0)$ such that

$$\frac{1}{\Lambda \left(f_0 - \varepsilon \right)} < \lambda < \frac{2 + \alpha\beta - 2\beta}{\left(2 + \alpha\beta - \beta \right) \left(f^\infty + \varepsilon \right) p^*}.$$

From the definition of f_0, it follows that there exists $\delta_1 > 0$ such that when $0 \leq u(t) \leq \delta_1$, for all $t \in I$, we have

$$f\left(u(t)\right) > \left(f_0 - \varepsilon\right) u\left(t\right) \qquad \text{for all } t \in I.$$

Let $\Omega_{\delta_1} = \{u \in K_1 \, : \, \|u\| < \delta_1\}$ and choose $u \in \partial\Omega_{\delta_1}$.

Since $p(s)\, G(1,s) \geq 0$ for all $s \in I$ and $u \in K_1$, we have

$$
\begin{aligned}
Tu\,(1) &= \lambda \int_0^1 G\,(1,s)\; p(s)\, f\,(u\,(s))\, ds \\
&\geq \lambda\,(f_0 - \varepsilon) \int_0^1 p(s)\, G\,(1,s)\; u\,(s)\; ds \\
&\geq \lambda\,(f_0 - \varepsilon)\, \|u\|_\infty \int_0^1 p(s)\, G\,(1,s)\; g\,(s)\, ds \\
&= \lambda\,(f_0 - \varepsilon)\; u(1)\, \Lambda \\
&> u(1).
\end{aligned}
$$

Thus, we have that $Tu(t) \leq u(t)$ is not true for all $t \in I$, which is a necessary condition to have $u - Tu \in K \subset K_1$.

Denoting by $\preceq$ the order induced by the cone K_1, we prove that $Tu \npreceq u$ and we deduce that

$$i_{K_1}(T, \Omega_{\delta_1}) = 0.$$

The arguments for $f_0 = +\infty$ are similar.

On the other hand, due to the definition of f^∞, we know that there exists $\delta_2 > \delta_1 > 0$ such that when $\min_{t \in I} \{u(t)\} \geq \delta_2$,

$$f(u(t)) \leq (f^\infty + \varepsilon)\, u(t) \leq (f^\infty + \varepsilon) \|u\|_\infty \quad \text{for all } t \in I.$$

Define

$$\Omega_{\delta_2} = \left\{ u \in K_1 : \min_{t \in I} |u(t)| < \delta_2 \right\}.$$

Ω_{δ_2} is an unbounded subset of the cone K_1.

Because of this, the fixed point index of the operator T with respect to Ω_{δ_2}, $i_{K_1}(T, \Omega_{\delta_2})$ is only defined in the case that the set of fixed points of the operator T in Ω_{δ_2} is compact and does not intersect $\partial\Omega_{\delta_2}$.

Let $u \in \partial\Omega_{\delta_2}$.

It is not difficult to verify that, for this range of values of the parameter λ, it holds that $\|T\,u\|_\infty < \|u\|_\infty$.

Thus $T\,u \neq u$ for all $u \in \partial\Omega_{\delta_2}$.

If $(I - T)^{-1}(\{0\}) \cap \Omega_{\delta_2}$ is unbounded we have infinite fixed points of T in Ω_{δ_2} and, as a consequence, Problem (1)-(2) has an infinite number of positive solutions on Ω_{δ_2} too.

In other case, from the fact that operator T is completely continuous and the set $(I - T)^{-1}(\{0\}) \cap \Omega_{\delta_2}$ is bounded and closed, it is not difficult to deduce that this set is equicontinuous in $C^1(I)$ and, as a consequence, compact.

In this last situation we may deduce that $\|Tu\| < \|u\|$ for all $u \in \partial\Omega_{\delta_2}$ and, as a consequence, we have that

$$i_{K_1}\left(T, \Omega_{\delta_2}\right) = 1.$$

Thus, we conclude that T has a fixed point in $\Omega_{\delta_2} \setminus \overline{\Omega}_{\delta_1}$, which is a positive solution of Problem (1)-(2). □

Corollary

Assume that condition (F) holds. Then,

(i) If $f_0 = +\infty$ *and* $f^\infty = 0$, *then for all* $\lambda > 0$ *Problem* (1)-(2) *has at least one positive solution.*

(ii) If $f_0 = +\infty$ *and* $0 < f^\infty < +\infty$, *then for all* $\lambda \in \left(0, \frac{2+\alpha\beta-2\beta}{(2+\alpha\beta-\beta)f^\infty p^*}\right)$ *Problem* (1)-(2) *has at least one positive solution.*

(iii) If $0 < f_0 < +\infty$ *and* $f^\infty = 0$, *then for all* $\lambda > \frac{1}{\Lambda f_0}$ *Problem* (1)-(2) *has at least one positive solution.*

Alternative existence results are deduced by considering the sets

$$K_\rho = \{u \in K_1 : \|u\| < \rho\}.$$

Lemma

Denote

$$f^\rho = \text{sup ess} \left\{ \frac{|p(t)|\ f(u)}{\rho};\ (t,u) \in I \times [0,\rho] \right\}.$$

If there exists $\rho > 0$ *such that* $\lambda f^\rho < \frac{2+\alpha\beta-2\beta}{2+\alpha\beta-\beta}$, *then*

$$i_{K_1}(T, K_\rho) = 1.$$

Lemma

Let

$$M = \left(\int_0^1 |G(1,s)|\, ds \right)^{-1}$$

and

$$f_\rho = inf\ ess \left\{ \frac{|p(t)|\, f(u)}{\rho};\ (t,u) \in I \times [0,\rho] \right\}.$$

If there exists $\rho > 0$ *such that* $\lambda f_\rho > M$, *then*

$$i_{K_1}(T, K_\rho) = 0.$$

THEOREM (2)

Assume $0 < \eta < 1/3$. *Then Problem* (1)–(2) *has at least one nontrivial solution in* K_1 *if one of the following conditions hold.*

(C1) There exist $0 < \rho_1 < \rho_2$, *such that* $\lambda f_{\rho_1} > M$ *and* $\lambda f^{\rho_2} < \frac{2+\alpha\beta-2\beta}{2+\alpha\beta-\beta}$.

(C2) There exist $0 < \rho_1 < \rho_2$, *such that* $\lambda f^{\rho_1} < \frac{2+\alpha\beta-2\beta}{2+\alpha\beta-\beta}$ *and* $\lambda f_{\rho_2} > M$.

Theorem (3)

Let $[a, b] \subset I$, with $a > 0$, be given. If one of the following conditions holds

(i) $f(x) < m^* x$ for every $x \geq 0$, where

$$m^* = \left(\lambda \sup_{t \in I} \int_0^1 G(t, s)\, p(s)\, ds \right)^{-1}.$$

(ii) $f(x) > m_* x$ for every $x \geq 0$, where

$$m_* = \left(\lambda \inf_{t \in [a,b]} \int_a^b G(t, s)\, p(s)\, ds \right)^{-1}.$$

Then Problem (1)–(2) has not nontrivial solution in K_1.

Example

Let $0 \leq \alpha \leq 1, 0 \leq \beta < \frac{2}{2-\alpha}$ and $0 \leq \eta \leq \frac{1}{3}$, and consider the problem

$$\begin{aligned} u''' &= -\lambda u^{\gamma} q(t) \arctan(t-\eta), \; t \in I, \\ u(0) &= 0, \; u''(\eta) = \alpha u'(1), \; u'(1) = \beta u(1), \end{aligned}$$

with $\gamma \in (0,1)$ and $c_1 \geq q(t) \geq c_2 > 0$ for all $t \in I$.

In this case,

$$f_0 = +\infty \text{ and } f^{\infty} = 0.$$

From **Theorem (1)** there exists at least one positive solution for all $\lambda > 0$.

Example

On the other hand, for $\rho > 0$,

$$f_\rho = \inf \text{ess} \left\{ \frac{q(t) \arctan |t-\eta| \; u^\gamma}{\rho}; \; (t,u) \in I \times [0,\rho] \right\} = 0$$

and it is not possible to find a positive ρ, such that $\lambda f_\rho > M$, which means that **Theorem (2)** can not be applied in this case.

Example

Let $0 \leq \alpha \leq 1, 0 \leq \beta < \frac{2}{2-\alpha}$ and $0 \leq \eta \leq \frac{1}{3}$, and consider the problem

$$\begin{aligned} u''' &= -\lambda u q(u) \arctan(t-\eta), \ t \in I, \\ u(0) &= 0, \ u''(\eta) = \alpha u'(1), \ u'(1) = \beta u(1), \end{aligned}$$

with $D > q(u) \geq c > 0$ for all $t \in I$ where

$$D \equiv \frac{2+\alpha\beta-2\beta}{\lambda\left(1-\eta^2\right)\left(\left(-\frac{1}{2}\ln\left(((\eta-2)\eta+2)\left(\eta^2+1\right)\right)-(\eta-1)\arctan(1-\eta)+\eta\arctan\eta\right)\right)}.$$

Example

Since,

$$\frac{1}{m^*} = \lambda \,\text{sup ess}_{\,t\in I} \int_0^1 G(t,s)\, p(s)\, ds$$
$$\leq \lambda \max_{t,s\in I} |G(t,s)| \int_0^1 |\arctan(s-\eta)| ds = \frac{1}{D},$$

then, $f(u) = uq(u) < u\,D = u\,m^*$.

Theorem (3) ensures that the considered problem has not nontrivial solutions in K_1.

THANKS FOR YOUR ATTENTION