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Strongly nonlinear multiplicative inequalities and applications to PDEs

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Outlines

1. History - Kolmogorov, Gagliardo- Nirenberg inequalities and strongly nonlinear variants
2. Inequalities without “weight” (Orlicz setting, together with Katarzyna Pietruska-Paluba)
3. Inequalities involving “weight $h(\cdot)$” in dimension 1, with Lebesque measure, and applications (together with Jan Peszek)
   - The special case
   - Generalization to Orlicz spaces
4. n dimensional case (with Tomasz Choczewski and daImil Pesa and Tomas Roskovec)
5. Inequalities involving nonlocal operators (with Claudia Capogne and Alberto Fiorenza)

- The presentation is based on a series of joint works with:

  Katarzyna Pietruska-Pałuba,
  Jan Peszek,
  Katarzyna Mazowiecka,
  Tomasz Choczewski,
  Ignacy Lipka,
  Aberto Fiorenza and Claudia Capogne,
  Tomas Roskovec and Dalmil Pesa.

Inequalities of Kolmogorov and of Gagliardo and Nirenberg

Theorem (Kolmogorov, 1939)

Zachodzi nierówność:

$$\|f^{(k)}\|_\infty \leq C_{k,m}\|f\|_\infty^{1-\frac{k}{m}}\|f^{(m)}\|_\infty^{\frac{k}{m}},$$

where f *is definef on* $\mathbb{R}$, $0 < k < m$, $k, m \in \mathrm{N}$.

Theorem (E. Gagliardo, D. Nirenberg, 1959)

There holds:

$$\|\nabla^{(k)} f\|_{L^p(\Omega)} \leq C\|f\|_{L^q(\Omega)}^{1-\frac{k}{m}} \|\nabla^{(m)} f\|_{L^r(\Omega)}^{\frac{k}{m}} + \|f\|_{L^q(\Omega)},$$

where $f : \Omega \to \mathrm{R}$, Ω*-regular enough,* $0 < k < m$, $k, m \in \mathrm{N}$, *and*

$$\frac{1}{p} = \frac{1-k}{m}\frac{1}{q} + \frac{k}{m}\frac{1}{r}.$$

Strongy nonlinear inequaalities

We are interested in inequality (SNMI):[1]

$$\int_{(a,b)\cap\{f>0\}} |f^{'}(x)|^p h(f(x))dx \leq C \int_{(a,b)\cap\{f>0\}} \left(\sqrt{|f''(x)\mathcal{T}_h(f(x))|}\right)^p h(f(x))dx, \quad (1)$$

and its generalizations.

[1] we propose its name: SNMI="Strongly nonlinear multiplicative nequality"

Assumptions (in most cases):

- $-\infty \le a < b \le +\infty$, $p \ge 2$,
- $f \in \mathcal{R}$, $C_0^\infty(a,b) \subseteq \mathcal{R} \subseteq W^{2,1}_{loc}(a,b)$, $f \ge 0$ (in most cases),
- $h : (0,\infty) \to [0,\infty)$ is continuous,
- $\mathcal{T}_h(\cdot)$ is continuous, interpreted as the transformation of f:

$$\mathcal{T}_h(\lambda) := \begin{cases} \frac{H(\lambda)}{h(\lambda)} & \text{if} \quad h(\lambda) \neq 0, \\ 0 & \text{if} \quad h(\lambda) = 0, \end{cases}$$

where H is primitive to h. Important property: when $h(\lambda) = \lambda^\alpha$ then $\mathcal{T}_h(\lambda) \sim \lambda$ and inequality is of the form:

$$\int_{(a,b)\cap\{f>0\}} |f^{'}(x)|^p h(f(x))dx \le C \int_{(a,b)\cap\{f>0\}} \left(\sqrt{|f''(x)f(x)|}\right)^p h(f(x))dx$$

[2]

[2] $\mathcal{T}_h(\lambda) := 1/\left(\ln H(\lambda)\right)^{'}$ if $H(\lambda) > 0$

Our motivations:

- Apply the new inequalities to regularity theory in singular PDEs in the form

$$\Delta u = g(x)\tau(u), \quad g \in L^p(\Omega),\ \Omega \subseteq \mathrm{R}^n$$

- and more general ones

$$Pu = g(x)\tau(u), \quad g \in L^p(\Omega),\ \Omega \subseteq \mathrm{R}^n,$$

where P is elliptic operator.

Example models typical for applications

- Thomas-Fermi model (1927, describes electric charge in isolated neutral atom)
$$\begin{cases} y''(t) = t^{\frac{1}{2}} y(t)^{\frac{3}{2}}, \ t \in (0,\infty), \\ y(0) = 0, \lim_{t\to\infty} y(t) = 0. \end{cases}$$
- Emden-Fowler problem (fluid dynamics):
$$\begin{cases} y'' + \lambda q(x) y^{-\gamma} = 0, \ x \in (0,1), \gamma > 0 \\ y(0) = y(1) = 0, \end{cases}$$
- model of membrana and model of mikro-electro-mechanical system (MEMS), papers by Esposito and coauthors
$$\begin{cases} -\Delta u = \frac{\lambda g(x)}{(1-u)^2} & \text{in} & \Omega \subseteq \mathrm{R}^2 \\ 0 \le u < 1 & \text{in} & \Omega \\ u = 0 & \text{on} & \partial\Omega \end{cases}$$
- models in cosmology, e.g. Makutuma model
$$\Delta u + \frac{1}{1+|x|^2} u^q = 0, \ x \in \mathrm{R}.$$

Unweighed simplest variant - multidimensional setting

Theorem (Katarzyna Pietruska-Paluba and A.K, 2006)

$$\int G(|\nabla u|)dx \ \leq \ C\int G(\sqrt{|u||\nabla^{(2)}u|})dx, \tag{2}$$

where G is convex, $G(\lambda)/\lambda^2$ is bouded near 0.

Inequalities involving "weight" in dimension d=1

Theorem (Jan Peszek and A.K., 2012)

$$\int_{(a,b)\cap\{f>0\}} |f^{'}(x)|^p h(f(x))dx \leq$$

$$C\int_{(a,b)\cap\{f>0\}} \left(\sqrt{|f''(x)\mathcal{T}_h(f(x))|}\right)^p h(f(x))dx,$$

under certain assumptions on h and f.

The special case: $h(\lambda) = \lambda^{-\theta p}$

Theorem (Jan Peszek and A.K., 2012)

Let $2 \leq p < \infty$, $\theta \in \mathbb{R}$ and $f \in W^{2,1}_{loc}(\mathbb{R})$ be such that $f^{'}$ has compact support. Assume additionally that at least one of the conditions is satisfied:

1. $\theta < \frac{1}{p}$,
2. $\theta > \frac{1}{p}$ *and f is nonnegative or (more generally) does not have isolated zeroes,*
3. $\theta > \frac{1}{p}$ *and there exists ϵ such that for every $r < R$:*

$$\int_{(r,R)\cap\{x:0<|f(x)|<\epsilon\}} \left(\frac{|f^{'}|}{|f|^{\theta}} \right)^p dx < \infty.$$

Then

$$\int_{\{x:f(x)\neq 0\}} \left(\frac{|f'|}{|f|^{\theta}}\right)^p dx \leq$$

$$\left(\frac{p-1}{|1-\theta p|}\right)^{\frac{p}{2}} \int_{\{x:f(x)\neq 0\}} \left(\frac{\sqrt{|ff''|}}{|f|^{\theta}}\right)^p dx.$$

For $\theta = \frac{1}{2}$ and $f \geq 0$ we retrieve Mazja's inequality (book) know earlier.

Conjecture. Constant $\left(\frac{p-1}{|1-\theta p|}\right)^{\frac{p}{2}}$ is precise.

Generalization to Orlicz spaces

We consider certain set of assumptions:

(M) $M : [0,\infty) \to [0,\infty)$ is (convex) differentiable $N-$function, and M satisfies the condition:

$$d_M \frac{M(\lambda)}{\lambda} \leq M'(\lambda) \leq D_M \frac{M(\lambda)}{\lambda} \quad \text{for every } \lambda > 0, \tag{3}$$

where $D_M \geq d_M \geq 2$.

(h) $h : (0,\infty) \to (0,\infty)$, $|h^{'} H| < Eh^2$, E-small enough + some assumptions.

Theorem (Jan Peszek and A.K, 2013)

Assume that M satisfies **(M)**, $h : (0,\infty) \to (0,\infty)$ *satisfies* **(h)**. *Then any nonnegative $f \in W^{2,1}(\mathbb{R})$ such that $f^{'}$ has compact support satisfies inequality*

$$\int_{\mathbb{R}\cap\{f>0\}} M(|f^{'}(x)|h(f(x)))dx \leq C \int_{\mathbb{R}\cap\{f>0\}} M\left(\sqrt{|f''(x)\mathcal{T}_h(f(x))|} \cdot h(f(x))\right) dx.$$

Application to the capacitary estimates

Mazya used the inequality:

$$\int_{\{x:f(x)\neq 0\}} \left(\frac{|f'|}{\sqrt{f}}\right)^p dx \leq (2(p-1))^{\frac{p}{2}} \int_{\{x:f(x)\neq 0\}} \left(\frac{\sqrt{|ff''|}}{\sqrt{|f|}}\right)^p dx,$$

to obtain the capacitary inequality:

$$\int_\Omega \mathrm{cap}_p^+(\mathcal{N}_t,\Omega) t^{p-1} dt \leq C \int_\Omega |\nabla^{(2)} u(x)|^p dx,$$

where $\mathcal{N}_t = \{x \in \Omega : u(x) \geq t\}$,

$$\mathrm{cap}_p^+(E,\Omega) := \inf \left\{ \int_\Omega |\nabla^{(2)} u|^p dx : u \in C_0^\infty(\Omega), u \geq 0 \text{ on } \Omega, \right. \\ \left. u \equiv 1 \text{ in a neighborhood of } E \right\},$$

whenever $E \subseteq \Omega$ is compact.

Let

- μ is a given Borel measure defined on open set Ω,
- N be the given N-function,
- N^* be the Legendre transform of N,
- $L_N(\Omega, \mu)$ be an Orlicz space related to N.

Theorem (Mazya: book, Theorem 8.3.1)

The following statements (a) and (b) are equivalent:
(a) The embedding:

$$\| |u|^p \|_{L_N(\Omega,\mu)} \leq A \|\nabla^{(2)} u\|^p_{L^p(\Omega)} \tag{4}$$

holds for every nonnegative $u \in C_0^\infty(\Omega)$, with a u-independent finite constant A.
(b) The following isoperimetric inequality:

$$\mu(E)(N^*)^{-1}\left(\frac{1}{\mu(E)}\right) \leq B \mathrm{cap}_p^+(E,\Omega) \tag{5}$$

holds for every compact $E \subset \Omega$, such that $\mathrm{cap}_p^+(E,\Omega) > 0$. Moreover, if A and B are the best constants in (4) and (5), respectively, then $B \leq A \leq pBC$, where C is the same as in the capacitary estimate.

We asked about the validity of a more general embedding:

$$\|M(|u|)\|_{L_N(\Omega,\mu)} \leq A \int_{\Omega} M(|\nabla^{(2)} u|)dx, \tag{6}$$

where $u \in C_0^\infty(\Omega)$ is nonnegative, with a (possibly) general convex function M instead of λ^p.

Theorem (Jan Peszek, A.K., 2013)

Under suitable assumptions on M, (6) is equivalent to the isoperimetric inequality:

$$\mu(E)(N^*)^{-1}\left(\frac{1}{\mu(E)}\right) \leq B\mathrm{cap}^+_M(E,\Omega), \tag{7}$$

holding over all compact sets $E \subset \Omega$ such that $\mathrm{cap}^+_M(E,\Omega) > 0$, where

$$\mathrm{cap}^+_M(E,\Omega) := \inf\left\{\int_\Omega M(|\nabla^{(2)}u|)dx : u \in C_0^\infty(\Omega), u \geq 0 \text{ on } \Omega, \right.$$
$$\left. u \equiv 1 \text{ in a neighborhood of } E\right\}.$$

- For that we only needed the SNMI with $h(\lambda) = \lambda^{-1/2}$ inside M, like in Mazja'a approach.

Applications to the nonlinear ODEs

Consider the following O.D.E:

$$\begin{cases} f''(x) = g(x)\tau(f(x)) \text{ a.e. in } (a,b), \\ f \in \mathcal{R} \end{cases} \tag{8}$$

where $-\infty \leq a < b \leq +\infty$ and:

- $\tau : A \to \mathbb{R}$, $A \subseteq [0,\infty)$ is an interval,
- $f \in W^{2,1}_{loc}((a,b))$, $f(x) \in A$,
- $g \in L^q(a,b)$, $q \in [1,\infty]$,
- set $\mathcal{R}$ defines the boundary conditions (ok for Dirichlet bc).

We find function $h(\cdot)$ such that[3]

$$|g(x)|^q = \left| \frac{f^{''}(x)}{\tau(f(x))} \right|^q = |\mathcal{T}_h(f(x))f^{''}(x)|^{\frac{2q}{2}} h(f(x)) = (*),$$

We apply:

$$\int_{(a,b)} |f^{'}(x)|^{2q} h(f(x))dx \leq$$

$$\left(\sqrt{2q-1}\right)^{2q} \int_{(a,b)} \left(\sqrt{|f''(x)\mathcal{T}_h(f(x))|}\right)^{2q} h(f(x))dx$$

$$= \int (*)dx = \int |g(x)|^q dx < \infty.$$

[3] ok when $|1/\tau(\lambda)|^q = |H(\lambda)/h(\lambda)|^q h(\lambda)$- we have to solve the ODE with the unknown H, $H^{'} = h$.

We deduce that

$$\int |f^{'}|^{2q} h(f) \leq C\|g\|_q^q.$$

- Let $G = G_\tau$ be such transform of τ that $|(G(f))^{'}|^{2q} = |f^{'}|^{2q} \cdot h(f)$ $(G^{'} = h^{1/(2q)})$. Then $G(f) \in W^{1,2q}((a,b))$, so is λ-Hölder continuous, where $\lambda = 1 - \frac{1}{2q}$.
- we deduce the regularity and asymptotic behavior of solutions.

Application (with Jan Peszek, generalization with Katarzyna Mazowiecka)

Assumption: $1 \leq q < \infty$, $\alpha \neq -1 + \frac{1}{q}$, $\kappa = -\mathrm{sign}(\alpha + 1 - \frac{1}{q})$, $0 < b \leq \infty$, $g \in L^q(0,b)$ and let $f \in W^{2,1}_{loc}(0,b)$ and $f \geq 0$ solves:

$$f^{''}(x) = g(x)(f(x))^{\alpha} \text{ a.e. on } (0,b)$$

and nonlinear boundary condition (mixed type):

$$\liminf_{R \nearrow b} \kappa |f^{'}(R)|^{2q-2} f^{'}(R)(f(R))^{-q(\alpha+1)+1}$$
$$- \limsup_{r \searrow 0} \kappa |f^{'}(r)|^{2q-2} f^{'}(r)(f(r))^{-q(\alpha+1)+1} \leq 0.$$

[4]

[4] Dirichlet bc - ok.

Theorem (Jan Peszek, A.K., 2012)

i)

$$\int_0^b |f'(x)|^{2q}|f(x)|^{-q(\alpha+1)}dx \leq C_q \int_0^b |g(x)|^q dx,$$

ii)

$$\sup\left\{\frac{|(f(x))^{\frac{1-\alpha}{2}} - (f(y))^{\frac{1-\alpha}{2}}|}{|x-y|^{1-\frac{1}{2q}}} : x, y \in (0,b)\right\} \leq$$

$$A_q \left(\int_0^b |g(x)|^q dx\right)^{\frac{1}{2q}},$$

iii) *If* $\alpha < 1$ *then* $\lim_{r\searrow 0} f(r) =: f(0) = 0$ *then*

$$|f(x)|^{\frac{1-\alpha}{2}} \leq A_q |x|^{1-\frac{1}{2q}} \left(\int_0^b |g(x)|^q dx\right)^{\frac{1}{q}}.$$

Extensions were obtained with Katarzyna Mazowiecka (2015).

Questions about better regularity

- Can $G(f) \in W^{2,2q}((a,b))$? Answer: in gerenaral 'no' (with Katarzyna Mazowiecka);
- Can we expect $G(f) \in W^{s,2q}((a,b))$ with $1 < s < 2$? We expect such a phenomena but for that we need SNMI with ∇^s where $1 < s < 2$- open !!!.

Generalization to $d > 1$

We obtained the analogue of multiplicative inequality having the form:

$$\int_\Omega |\nabla f(x)|^p h(f(x))dx \leq$$
$$C \int_\Omega \left(\sqrt{|Pf(x)\mathcal{T}_h(f(x))|} \right)^p h(f(x))dx, \quad (9)$$

and applications to the eigenvalue problems like:

$$\begin{cases} \Delta f(x) &= g(x)\tau(f(x)) \text{ a.e. in } \Omega. \\ f \in \mathcal{R} & \end{cases} \quad (10)$$

where P- is the elliptic operator (with Dalmil Pesa and Tomas Roskovec), $P = \Delta$ - earlier with Tomasz Choczewski.
[5]

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[5] $f \in \mathcal{R}$ - OK for $f > 0$ in Ω, $f \equiv 0$ on $\partial\Omega$.

- for the choice of the admitted weights $h(\cdot)$ we require information about:
  - best constants in the Hardy inequality (for the radial case);
  - best constant in the inequality (for general case)

$$\left(\int_\Omega |\Delta^{\spadesuit} w|^q dx\right)^{\frac{1}{q}} \le D_{q,\Omega} \left(\int_\Omega |\Delta w|^q dx\right)^{\frac{1}{q}},$$

where $1 < q < \infty$,

$$\begin{aligned} \Delta^{\spadesuit} u(x) := & \quad \Delta_\infty u(x) - \Delta u(x), \\ \Delta_\infty u(x) = & \sum_{i,j\in\{1,\dots,n\}} v_i(x) v_j(x) \frac{\partial^{(2)} u}{\partial x_i \partial x_j}(x) \\ v(x) = & \quad \frac{\nabla u(x)}{|\nabla u(x)|} \chi_{\{\nabla u(x)\neq 0\}} \in \mathbb{R}^n. \end{aligned} ,$$

- SNMI applies to the model of elektrostatic micromechanical systems (MEMS), which is reduced to the following problem

$$\begin{cases} \Delta u = \frac{\lambda f(x)}{(1-u)^2} & w \quad \Omega \\ u = 0 & on \quad \partial\Omega \\ 0 < u < 1 & in \quad \Omega \end{cases}$$

where $\lambda \geq 0$, $f \geq 0, f \in L^q(\Omega)$, $u \in C^1(\overline{\Omega} \cap W^{2,2}(\Omega))$, and Ω is open and bounded (papers by Esposito).
In particular we get: $\sqrt{1-u} \in W^{1,2q}(\Omega)$ and

$$\int_\Omega |\nabla(\sqrt{1-u})|^{2q} dx \leq C\lambda^q \int_\Omega |f(x)|^q dx.$$

Weighted variants in 1-d (with Ignacy Lipka)

$$\int_{(a,b)} |f^{'}(x)|^p h(f(x))\rho(x)dx \leq$$

$$C\left(\int_{(a,b)} \left(\sqrt{|f''(x)\mathcal{T}_h(f(x))|}\right)^p h(f(x))\rho(x)dx\right.$$

$$\left. + \int_{(a,b)} |\mathcal{T}_h(f(x))|^p h(f(x))|\rho^{'}(x)|dx\right)$$

ρ satisfies B_p condition due to Kufner and Opic: $\rho^{-1/(p-1)} \in L^1_{loc}$, $\rho \in AC_{loc}$.

Inequalities involving nonlocal operators (with Claudia Capogne and Alberto Fiorenza)

Let $d_A > 1$ (lower Simonnenko index) and let $1 < p < i_A$ (Boyd index). We obtain inequalities:

i)

$$\int_{\mathrm{R}} A(|f^{'}(x)|)dx \leq C_{A,p} \int_{\mathrm{R}} A\left(\sqrt[p]{|f^{''}(x)\mathcal{T}_{h\equiv 1,p}(f,x)|} \right) dx.$$

For every nonnegative $f \in W^{2,1}_{loc}(\mathrm{R})$ such that $f^{'}$ is compactly supported;

Second example inequality

ii)

$$\int_{\mathsf{R}} M(|f^{'}(x)|h(f(x)))dx \leq$$

$$A\int_{\mathsf{R}} M\left(B\sqrt[p]{|\mathcal{M}f''(x)\mathcal{T}_{h,p}(f,x)|}\cdot h(f(x))\right)dx,$$

$$\mathcal{T}_{h,p}(f,x) := \begin{cases} \frac{\int_{-\infty}^{x} \Phi_p\left(h(f(y))f^{'}(y)\right)\chi_{\{f(y)\neq 0\}}\,dy}{h(f(x))^{p-1}}, & f(x)\neq 0 \\ 0, & f(x)=0 \end{cases}$$

and $\phi_p(s) = |s|^{p-2}s$. For $p = 2$ we have $\mathcal{T}_{h,p} = \mathcal{T}_h$, $\mathcal{M}h$ is Hardy - Littlewood maximal function.

Indeces

The approach requires analysis Simmonnenko and Boyed indeces and their invariances: if the inequality holds with comvex function M then it holds with $M_1 \sim M$.

- Simonnenko indeces for A (convex):

$$d_A := \inf_{t>0} \frac{tA^{'}(t)}{A(t)}, \quad D_A := \sup_{t>0} \frac{tA^{'}(t)}{A(t)}.$$

- Boyd indeces for A:

$$i_A := \sup_{A_1 \sim A} d_{A_1} \quad I_A := \inf_{A_1 \sim A} D_{A_1}.$$

Interpretation of the nonlinear transform

- For $h \equiv 1$ we have

$$\mathcal{T}_{h\equiv 1,p}(f,x) = \Delta^{-1}\Delta_p, \quad \Delta_p u = \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right).$$

- In general

$$\mathcal{T}_{h,p}(f,x) = \frac{\Delta^{-1}\Delta_p(H(f))}{\Phi_p(h(f))}, \quad \Phi_p(s) = |s|^{p-2}s.$$

- In particular $\mathcal{T}_{h,p}(f,x)$ is nonlocal.

Multidimensional variants - with Tomasz Choczewski (2018, 2019)

i)

$$\int_\Omega |\nabla u(x)|^p h(u(x))dx \le C(n,p) \int_\Omega \left(\sqrt{|\nabla^{(2)} u(x)||\mathcal{T}_h(u(x))|} \right)^p h(u(x))dx,$$

ii)

$$\int_\Omega |\nabla u(x)|^p h(u(x))dx \le D(n,p) \int_\Omega \left(\sqrt{|\Delta u(x)||\mathcal{T}_h(u(x))|} \right)^p h(u(x))dx,$$

$d > 1$ - with Tomas Roskovec and Dalmil Pesa (presently)

We work on generalisation of ii) to arbitrary operator elliptic opertor P (not necessarily with constant coefficients):

$$\int_\Omega |\nabla u(x)|^p h(u(x))dx \leq$$

$$D(n,p) \int_\Omega \left(\sqrt{|Pu(x)||\mathcal{T}_h(u(x))|} \right)^p h(u(x))dx,$$

The goal: elliptic regularity theory for solutions of:

$$Pu = g(x)\tau(u).$$

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Thank you very much for your attension.