On pairs of complementary boundary and transmission conditions

Elżbieta Ratajczyk & Adam Bobrowski

Lublin University of Technology, Poland

A curious result

Perpendicular boundary conditions

$$\begin{aligned} f'(0) &= \gamma f(0) \quad \perp \quad f''(0) = \gamma f'(0), \quad (\gamma > 0), \\ f'(0) &= 0 \quad \perp \quad f(0) = 0 \quad (\text{not } f''(0) = 0). \end{aligned}$$

A curious result

Perpendicular boundary conditions

$$\begin{aligned} f'(0) &= \gamma f(0) \quad \perp \quad f''(0) = \gamma f'(0), \quad (\gamma > 0), \\ f'(0) &= 0 \quad \perp \quad f(0) = 0 \quad (\text{not } f''(0) = 0). \end{aligned}$$

Perpendicular transmission conditions

$$\begin{aligned} f'(0+) &= \gamma[f(0+) - f(0-)], \qquad f(0-) = -f(0+), \\ f'(0-) &= \delta[f(0+) - f(0-)] \quad \perp \quad f''(0+) = \delta f'(0-) + \gamma f'(0+). \end{aligned}$$

A curious result

Perpendicular boundary conditions

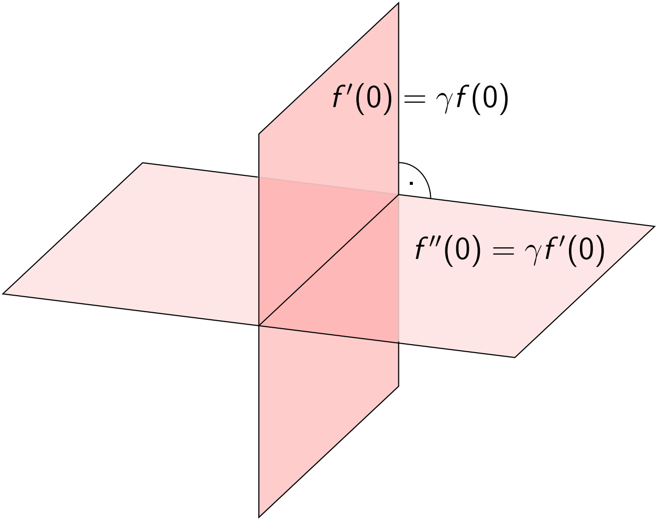
$$
\begin{aligned}
f'(0) &= \gamma f(0) & \perp \quad & f''(0) = \gamma f'(0), \quad (\gamma > 0),\\
f'(0) &= 0 & \perp \quad & f(0) = 0 \quad (\text{not } f''(0) = 0).
\end{aligned}
$$

Perpendicular transmission conditions

$$
\begin{aligned}
f'(0+) &= \gamma[f(0+) - f(0-)], & & f(0-) = -f(0+),\\
f'(0-) &= \delta[f(0+) - f(0-)] & \perp \quad & f''(0+) = \delta f'(0-) + \gamma f'(0+).
\end{aligned}
$$

Meaning, probabilistic interpetation?

In a certain Banach space:



Semigroups: first order Cauchy problems

- Banach space $\mathbb{F}$;
- operator $A : \mathbb{F} \supset D(A) \to \mathbb{F}$,
- search for solution u of

$$u'(t) = Au(t), \qquad u(0) = f \in \mathbb{F}.$$

Semigroups: first order Cauchy problems

- Banach space $\mathbb{F}$;
- operator $A : \mathbb{F} \supset D(A) \to \mathbb{F}$,
- search for solution u of

$$u'(t) = Au(t), \qquad u(0) = f \in \mathbb{F}.$$

- Well-posed (existence, uniqueness and continuous dependence on initial data) iff A a semigroup generator;
- meaningful $\{\mathrm{e}^{tA}, t \geqslant 0\}$ – family of bounded linear operators in $\mathbb{F}$ such that $\mathrm{e}^{sA}\mathrm{e}^{tA} = \mathrm{e}^{(s+t)A}, s, t \geqslant 0$, $\lim_{t\to 0} \mathrm{e}^{tA} f = f$.

Semigroups: examples

- A – bounded, $\mathrm{e}^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n, (t \in \mathbb{R})$,
- A – shift by 1: $\mathrm{e}^{tA} = \mathrm{e}^{t} E\, f(x + N(t))$, N – Poisson process,
- $A = \frac{\mathrm{d}}{\mathrm{d}x}$, $\mathrm{e}^{tA} f(x) = f(x + t)$,
- $A = \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $\mathrm{e}^{tA} f(x) = E\, f(x + w(t))$, w – Wiener process,
- $A = \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $D(A) \sim f'(0) = 0$, $\mathrm{e}^{tA} f(x) = E\, f(|x + w(t)|)$.

Cosines: second order Cauchy problems

- Search for solution u of

$$u''(t) = Au(t), \qquad u'(0) = 0, u(0) = f \in \mathbb{F}.$$

- Problem well-posed (existence, uniqueness and continuous dependence on initial data) iff A a cosine family generator;
- $C_A(t), t \in \mathbb{R}$ – family of bounded linear operators in $\mathbb{F}$ such that

$$2C_A(t)C_A(s) = C_A(t-s) + C_A(t+s), \qquad s, t \in \mathbb{R}$$

$\lim_{t\to 0} C_A(t)f = f$.

Cosines: second order Cauchy problems

- Search for solution u of

$$u''(t) = Au(t), \qquad u'(0) = 0, u(0) = f \in \mathbb{F}.$$

- Problem well-posed (existence, uniqueness and continuous dependence on initial data) iff A a cosine family generator;
- $C_A(t), t \in \mathbb{R}$ – family of bounded linear operators in $\mathbb{F}$ such that

$$2C_A(t)C_A(s) = C_A(t-s) + C_A(t+s), \qquad s,t \in \mathbb{R}$$

  $\lim_{t\to 0} C_A(t)f = f$.
- A – bounded, $C_A(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} A^{2n}$,
- $A = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $C_A f(x) = \frac{1}{2}[f(x+t) + f(x-t)], t, x \in \mathbb{R}$.

Cosine family generator is a semigroup generator

Weierstrass Formula:

$$\mathrm{e}^{tA}f = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{s^2}{4t}} C_A(s) f \,\mathrm{d}s, \qquad t > 0$$

Cosine family generator is a semigroup generator

Weierstrass Formula:

$$\mathrm{e}^{tA}f = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{s^2}{4t}} C_A(s)f \,\mathrm{d}s, \qquad t > 0$$

Additionally: the semigroup generated by a cosine family generator is ‘more regular’.

Three points of view

(A) Solutions to DE.

(B) Semigroup/cosine family.

(C) Operator A.

Three points of view

(A) Solutions to DE.

(B) Semigroup/cosine family.

(C) Operator A.

Described: (A) $\longleftrightarrow$ (B). To do: (B) $\longleftrightarrow$ (C).

From semigroup/cosine families to generators

- Given $\{T(t), t \geqslant 0\}$ such that $T(t+s) = T(s)T(s), s, t \geqslant 0$ and $\lim_{t\to 0} T(t)f = f$, we define

$$\boxed{Af = \lim_{t\to 0} t^{-1}(T(t)f - f)}$$

  for f (composing A's domain), for which this limit exists.
- Given $\{C(t), t \in \mathbb{R}\}$ s.t. $2C(s)C(s) = C(t+s) + C(t-s)$, $s, t \in \mathbb{R}$ and $\lim_{t\to 0} C(t)f = f$, we define

$$\boxed{Af = \lim_{t\to 0} 2t^{-2}(C(t)f - f)}$$

  for f (composing A's domain), for which this limit exists.

Generation theorem for semigroups

Hille–Yosida–F–P–M Theorem

The exponent of A exists and satisfies $\|e^{tA}\| \leqslant M\mathrm{e}^{\omega t}$, $t \geqslant 0$ iff A is closed and densely defined, and

$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}(\lambda - A)^{-1} \right\| \leqslant \frac{Mn!}{(\lambda - \omega)^{n+1}}, \qquad n \geqslant 0, \lambda > \omega$$

Generation theorem for semigroups

Hille–Yosida–F–P–M Theorem

The exponent of A exists and satisfies $\|e^{tA}\| \leqslant M\mathrm{e}^{\omega t}, t \geqslant 0$ iff A is closed and densely defined, and

$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} (\lambda - A)^{-1} \right\| \leqslant \frac{Mn!}{(\lambda - \omega)^{n+1}}, \qquad n \geqslant 0, \lambda > \omega$$

The last condition simplifies to

$$\left\| (\lambda - A)^{-n} \right\| \leqslant \frac{M}{(\lambda - \omega)^n}, \qquad n \geqslant 0, \lambda > \omega$$

and for $M = 1$ and $\omega = 0$, to

$$\|\lambda(\lambda - A)^{-1}\| \leqslant 1.$$

Checking this is from time to time doable — stochastic processes.

Generation theorem for cosine families

Sova and Da Prato–Giusti Theorem

A generates a cosine family which satisfies $\|C_A(t)\| \leqslant M\mathrm{e}^{\omega|t|}$, $t \in \mathbb{R}$ iff A is closed and densely defined, and

$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} [\lambda(\lambda^2 - A)^{-1}] \right\| \leqslant \frac{Mn!}{(\lambda - \omega)^{n+1}}, \qquad n \geqslant 0, \lambda > \omega$$

Generation theorem for cosine families

Sova and Da Prato–Giusti Theorem

A generates a cosine family which satisfies $\|C_A(t)\| \leqslant M\mathrm{e}^{\omega|t|}, t \in \mathbb{R}$ iff A is closed and densely defined, and

$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}[\lambda(\lambda^2 - A)^{-1}] \right\| \leqslant \frac{Mn!}{(\lambda-\omega)^{n+1}}, \qquad n \geqslant 0, \lambda > \omega$$

The last condition DOES NOT simplify, and checking it is almost never doable.
Hence, need for other methods.

Subspace semigroups/cosine families

- Given: a semigroup $\{T(t), t \geqslant 0\}$ in $\mathbb{F}$ with generator A
- Assumption: $\mathbb{F}_0 \subset \mathbb{F}$ is left invariant
- Then $\{T(t)_{|\mathbb{F}_0}, t \geqslant 0\}$ is a semigroup in $\mathbb{F}_0$ with generator

$$A_0 := A_{|D(A)\cap\mathbb{F}_0}.$$

Isomorphic/similar semigroups/cosine families

- Isomorphic Banach spaces $\mathbb{F}$ and $\mathbb{G}$: isomorphism $\mathcal{I} : \mathbb{F} \to \mathbb{G}$,
- Generator A of a semigroup in $\mathbb{F}$;

Isomorphic/similar semigroups/cosine families

- Isomorphic Banach spaces $\mathbb{F}$ and $\mathbb{G}$: isomorphism $\mathcal{I} : \mathbb{F} \to \mathbb{G}$,
- Generator A of a semigroup in $\mathbb{F}$;
- Then $\{\mathcal{I}\mathrm{e}^{tA}\mathcal{I}^{-1}, t \geqslant 0\}$ – a semigroup in $\mathbb{G}$;
- Its generator is 'the image of A in $\mathbb{G}$':

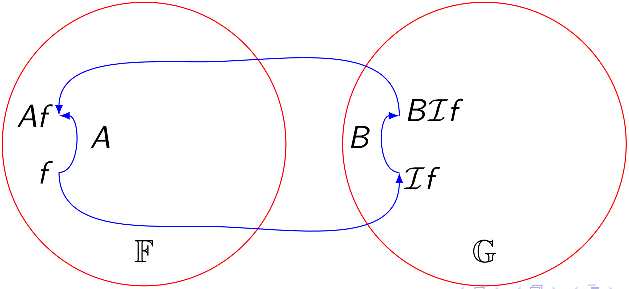
$$D(B) = \mathcal{I}D(A) \quad \text{and} \quad B\mathcal{I}f = \mathcal{I}Af, f \in D(A).$$

Isomorphic/similar semigroups/cosine families

- Isomorphic Banach spaces $\mathbb{F}$ and $\mathbb{G}$: isomorphism $\mathcal{I} : \mathbb{F} \to \mathbb{G}$,
- Generator A of a semigroup in $\mathbb{F}$;
- Then $\{\mathcal{I}\mathrm{e}^{tA}\mathcal{I}^{-1}, t \geqslant 0\}$ – a semigroup in $\mathbb{G}$;
- Its generator is 'the image of A in $\mathbb{G}$':

$$D(B) = \mathcal{I}D(A) \quad \text{and} \quad B\mathcal{I}f = \mathcal{I}Af, f \in D(A).$$

(Solving the heat equation by Fourier series ...)



Feller boundary conditions for 1D Laplace operator

Work in $C[0,\infty]$:

Candidates for generators:

$Af = f''$ for twice continuously differentiable functions with $f'' \in C[0,\infty]$, satisfying Feller b.c.

$$\alpha f''(0) - \beta f'(0) + \gamma f(0) - \delta \int_{\mathbb{R}_*^+} f \, \mathrm{d}\mu = 0,$$

where μ – a probability measure on $\mathbb{R}_*^+ = (0,\infty)$, and α, β, γ and δ – non-negative constants with $\gamma \geqslant \delta$ and $\alpha + \beta > 0$.

Interpretation of constants. Traditional approach – obstacles.

Solving differential equations ...

George (György) Pólya

In order to solve a differential equation you look at it till a solution occurs to you

Lord Kelvin's method of images

Idea:

use what you know from a larger space.

Use basic cosine family in $C[-\infty, \infty]$,

$$C(t)f(x) = \frac{1}{2}\left(f(x+t) + f(x-t)\right), \qquad x, t \in \mathbb{R}.$$

generated by $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$.

Lord Kelvin's method of images

Idea:

use what you know from a larger space.

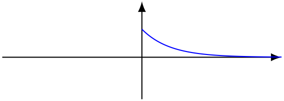
Use basic cosine family in $C[-\infty, \infty]$,

$$C(t)f(x) = \frac{1}{2}\left(f(x+t) + f(x-t)\right), \qquad x, t \in \mathbb{R}.$$

generated by $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$.
Find a subspace of $C[-\infty, \infty]$ that is invariant under the basic cosine family and isomorphic to $C[0, \infty]$.

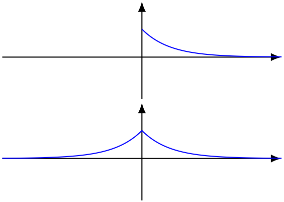
How it works for Neumann b.c. (Feller 1970)

Neumann boundary condition: $f'(0) = 0$.



How it works for Neumann b.c. (Feller 1970)

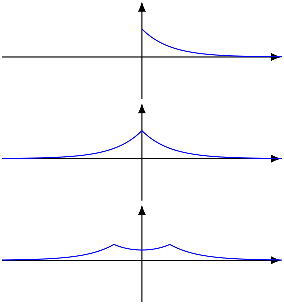
Neumann boundary condition: $f'(0) = 0$.



Extend: $\widetilde{f}(-x) = f(x)$

How it works for Neumann b.c. (Feller 1970)

Neumann boundary condition: $f'(0) = 0$.

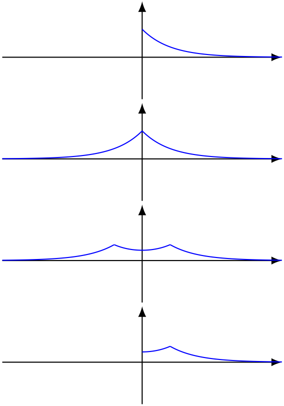


Extend: $\widetilde{f}(-x) = f(x)$

Use $C(t)$: $C(t)\widetilde{f}(x) = \frac{1}{2}[\widetilde{f}(x+t) + \widetilde{f}(x-t)]$

How it works for Neumann b.c. (Feller 1970)

Neumann boundary condition: $f'(0) = 0$.



Extend: $\widetilde{f}(-x) = f(x)$

Use $C(t)$: $C(t)\widetilde{f}(x) = \frac{1}{2}[\widetilde{f}(x+t) + \widetilde{f}(x-t)]$

Restrict to $x \geqslant 0$.

Abstract Kelvin Formula:

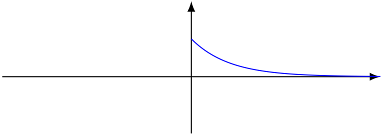
$$C_N(t) = RC(t)Ef.$$

Side notes:

- $2t^{-2}(C_N(t)f - f) = 2Rt^{-2}(C(t)Ef - Ef)$.
- For $f \in C^2[0,\infty]$, its even extension belongs to $C^2[-\infty,\infty]$ iff $f'(0) = 0$.
- Operator R — obvious. Operator E — not so obvious.
- How to connect a b.c. with extension operator?

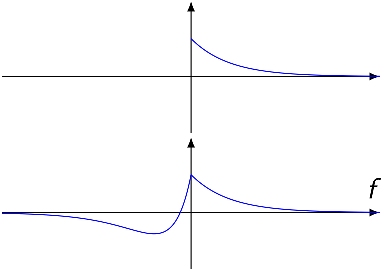
Other b.c.

- Dirichlet b.c.? Same procedure – just use odd extensions.
- Robin boundary condition: $f'(0) = \gamma f(0)$? Same. Different extension.



Other b.c.

- Dirichlet b.c.? Same procedure – just use odd extensions.
- Robin boundary condition: $f'(0) = \gamma f(0)$? Same. Different extension.



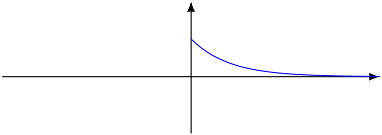
$$f(-x) = f(x) - 2\gamma \int_0^x \mathrm{e}^{-\gamma(x-y)} f(y)\,\mathrm{d}y, x \geqslant 0$$

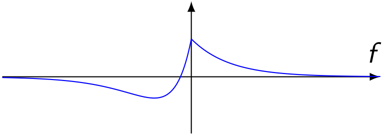
Abstract Kelvin Formula:

$$C_\gamma(t) = RC(t)Ef.$$

Other b.c.

- Dirichlet b.c.? Same procedure – just use odd extensions.
- Robin boundary condition: $f'(0) = \gamma f(0)$? Same. Different extension.





$$f(-x) = f(x) - 2\gamma \int_0^x \mathrm{e}^{-\gamma(x-y)} f(y)\,\mathrm{d}y, x \geqslant 0$$

Abstract Kelvin Formula:

$$C_\gamma(t) = RC(t)Ef.$$

Works for all b.c. presented above (A.B. circa 2010)

Lord Kelvin's method of images

Given

- basic cosine family in $C[-\infty,\infty]$ with generator $A = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$
- $\mathbb{F}_0 \subset C[-\infty,\infty]$ invariant;
  cosine family in $\mathbb{F}_0$ has generator $A_0 = A_{|D(A)\cap\mathbb{F}_0}$
- isomorphism $E\colon C[0,\infty] \to \mathbb{F}_0$, $E^{-1} = R$

Lord Kelvin's method of images

Given

- basic cosine family in $C[-\infty, \infty]$ with generator $A = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$
- $\mathbb{F}_0 \subset C[-\infty, \infty]$ invariant;
  cosine family in $\mathbb{F}_0$ has generator $A_0 = A_{|D(A)\cap\mathbb{F}_0}$
- isomorphism $E\colon C[0, \infty] \to \mathbb{F}_0$, $E^{-1} = R$

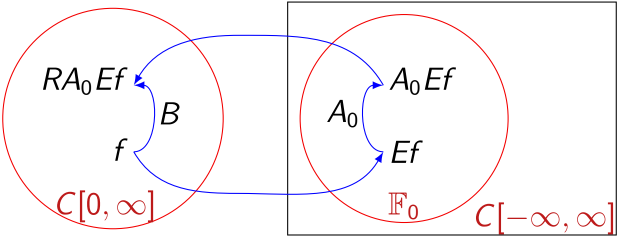
Lord Kelvin's method of images

Given

- basic cosine family in $C[-\infty,\infty]$ with generator $A = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$
- $\mathbb{F}_0 \subset C[-\infty,\infty]$ invariant;
  cosine family in $\mathbb{F}_0$ has generator $A_0 = A_{|D(A)\cap\mathbb{F}_0}$
- isomorphism $E\colon C[0,\infty] \to \mathbb{F}_0$, $E^{-1} = R$

Result: the cosine family in $C[0,\infty]$ generated by

$$B = RA_0E \quad \text{on the domain} \quad D(B) = RD(A_0).$$



Why this approach works?

Key ingredients:

1. Invariant subspace of $C[-\infty,\infty]$ which is isomorphic to $C[0,\infty]$;
2. The related extension operator E.
3. The invariant subspace intimately related to a b.c.

$$\textit{Boundary condition} \Longrightarrow \textit{invariant subspace}$$

Questions (June 2021)

1. $C^R_\gamma \subset C[-\infty,\infty]$ invariant for the basic cosine family
$$C^R_\gamma := \{f; f(-x) = f(x) - 2\gamma \int_0^x \mathrm{e}^{-\gamma(x-y)} f(y)\,\mathrm{d}y, x \geqslant 0\}.$$
2. Is C^R_γ complemented by another invariant subspace?
$$C[-\infty,\infty] = C_{\mathrm{even}} \oplus C_{\mathrm{odd}}; \qquad C[-\infty,\infty] = C^R_\gamma \oplus X \text{ ??????}$$
3. Can X be related to a boundary condition?

Questions (June 2021)

1. $C^R_\gamma \subset C[-\infty,\infty]$ invariant for the basic cosine family

$$C^R_\gamma := \{f; f(-x) = f(x) - 2\gamma \int_0^x \mathrm{e}^{-\gamma(x-y)} f(y)\,\mathrm{d}y, x \geqslant 0\}.$$

2. Is C^R_γ complemented by another invariant subspace?

$$C[-\infty,\infty] = C_{even} \oplus C_{odd}; \qquad C[-\infty,\infty] = C^R_\gamma \oplus X \text{ ??????}$$

3. Can X be related to a boundary condition?
4. How to meaningfully project onto C^R_γ? (The projection

$$Pf(x) = \begin{cases} f(x), & x \geqslant 0, \\ f(-x) - 2\gamma \int_0^{-x} \mathrm{e}^{-\gamma(-x-y)} f(y)\,\mathrm{d}y, & x < 0 \end{cases}$$

is not very informative ...)

An idea (mimic decomp. into even and odd parts):

Given $f \in C[-\infty, \infty]$, find for each $N > 0$, a $g \in C_\gamma^R$ that minimizes

$$\int_{-N}^{N} [f(x) - g(x)]^2 \, \mathrm{d}x.$$

An idea (mimic decomp. into even and odd parts):

Given $f \in C[-\infty, \infty]$, find for each $N > 0$, a $g \in C_\gamma^R$ that minimizes

$$\int_{-N}^{N} [f(x) - g(x)]^2 \, \mathrm{d}x.$$

This gives g of the form

$$g(x) = (g(0) - f(0))\mathrm{e}^{\gamma x} + f_{even}(x) + \gamma \int_0^x \mathrm{e}^{\gamma(x-y)} f(-y) \, \mathrm{d}y, x \in [0, N].$$

An idea (mimic decomp. into even and odd parts):

Given $f \in C[-\infty, \infty]$, find for each $N > 0$, a $g \in C^R_\gamma$ that minimizes

$$\int_{-N}^{N} [f(x) - g(x)]^2 \,\mathrm{d}x.$$

This gives g of the form

$$g(x) = (g(0) - f(0))\mathrm{e}^{\gamma x} + f_{even}(x) + \gamma \int_0^x \mathrm{e}^{\gamma(x-y)} f(-y)\,\mathrm{d}y, x \in [0, N].$$

Since we want (a) $\lim_{x\to\infty} g(x)$ to exist, and (b) $g \in C^R_\gamma$, this leads to

$$P_\gamma f(x) := g(x) := f_{even}(x) - \gamma \mathrm{e}^{\gamma x} \int_x^\infty \mathrm{e}^{-\gamma y} f(-y)\,\mathrm{d}y, \quad x \in \mathbb{R}.$$

Results:

Theorem 1

P_γ is a projection on C_γ^R

Theorem 2

$Q_\gamma := I - P_\gamma$ is a projection on

$$C_\gamma^F := \{f; f(-x) = -f(x) + 2\gamma \int_0^x \mathrm{e}^{-\gamma(x-y)} f(y)\,\mathrm{d}y + 2f(0)\mathrm{e}^{-\gamma x}, x \geqslant 0\}.$$

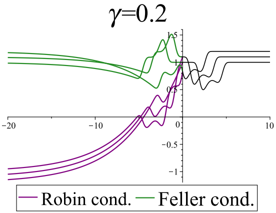
C_γ^F is related to the sticky boundary condition $f''(0) = \gamma f'(0)$ and in particular is invariant!

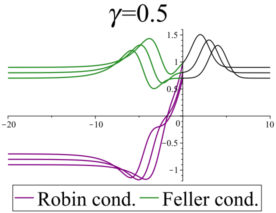
Theorem 3

$C[-\infty, \infty] = C_\gamma^R \oplus C_\gamma^F$, that is,

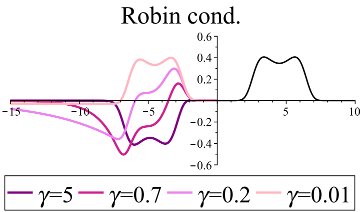
$$f'(0) = \gamma f(0) \qquad \perp \qquad f''(0) = \gamma f'(0).$$

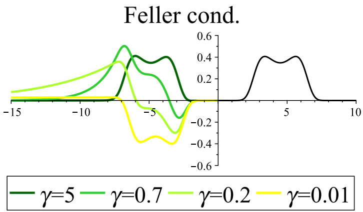
Examples of Robin and Feller extensions





Dependence on parameter γ:





About projections

N. L. Carothers 2004

outside of the Hilbert space setting, nontrivial projections are often hard to come by

$$
\begin{aligned}
P_\gamma f(x) &:= f_{even}(x) - \gamma \mathrm{e}^{\gamma x} \int_x^\infty \mathrm{e}^{-\gamma y} f(-y)\,\mathrm{d}y, \\
Q_\gamma f(x) &:= f_{odd}(x) + \gamma \mathrm{e}^{\gamma x} \int_x^\infty \mathrm{e}^{-\gamma y} f(-y)\,\mathrm{d}y, \qquad x \in \mathbb{R}.
\end{aligned}
$$

Properties of $\gamma \mapsto P_\gamma$

(a) It is continuous in the uniform operator topology for $\gamma \in (0, \infty)$.

(b) At $\gamma = \infty$: $\lim_{\gamma \to \infty} P_\gamma = P_\infty := Q_0$ in the strong topology.

(c) P_γ from projection on even to projection on odd; Q_γ in the other direction but their paths never cross!

Thinking of Ukraine

Thank you