A topological approach to Stieltjes Differential Equations

F. Adrián F. Tojo

CITMAga
Universidade de Santiago de Compostela

Universidade de Santiago de Compostela
Ministerio de Ciencia, Innovación y Universidades
Cofinanciado por la Unión Europea
Agencia Estatal de Investigación
CITMAGA Centro de Investigación y Tecnología Matemática de Galicia

… *We evidently must regard Time as passing with a steady flow;*
therefore it must be compared with some handy steady motion,
such as the motion of the stars, and especially of the Sun and the Moon;
such a comparison is generally accepted,
and was born adapted for the purpose by the Divine design of God.

Isaac Barrow
Lecture I: Generation of magnitudes (1664).

The idea

Since the time of Newton and Leibniz there has been the idea of deriving one variable with respect to another, regardless of which variables they are.

In 1894, Stieltjes introduced the analogous concept for integrals:

- *Derivative:*

$$f_g'(t)=\frac{\mathrm{d}\,f}{\mathrm{d}\,g}(t)=\lim_{s\to t}\frac{f(s)-f(t)}{g(s)-g(t)}\simeq\frac{\Delta f(t)}{\Delta g(t)}.$$

- *Riemann-Stieltjes Integral:*

$$\lim_{\{x_1,\dots,x_n\}}\sum_{i=1}^{n}f(c_i)\left[g(x_i)-g(x_{i-1})\right].$$

The *Lebesgue-Stieltjes integral* is defined as the integral with respect to the complete measure μ_g that satisfies $\mu_g([a,b))=g(b)-g(a)$.

Why use Stieltjes Calculus?

1. It allows to treat, with the same theoretical framework, differential equations, difference equations, equations with impulses and time scales.
2. It allows modeling complex phenomena where the dynamics has stationary parts (zones of constancy) and parts of sudden change (jumps).
3. It allows using measure theory to derive very powerful results.
4. Its development runs parallel to classical calculus (both continuous and discrete) using a very similar conceptual framework, so it is to be expected that classical results can be transferred to the Stieltjes case.
5. In contrast to the equations in measure, the Stieltjes calculus allows to obtain "classical" solutions (satisfying the equation at all points).

Young, Daniell, Petrovski, Liberman and others formalize in different ways the concept of derivative with respect to a function at the beginning of the 20th century, but the idea is not developed in detail.

A New Unification of Continuous, Discrete and Impulsive Calculus through Stieltjes Derivatives of Prof. Rodrigo López Pouso and Adrián Rodríguez appears in 2015.

Key ideas:

1. A definition of derivative for any left continuous non decreasing derivator.
2. The fundamental theorem of calculus (for Lebesgue-Stieltjes and Kurzweil-Stieltjes integrals).

From this point on, further research into the basic properties of the Stieltjes calculus begins.

Basic definitions

Let I be an interval. A non-decreasing and left continuous function $g : I \to \mathbb{R}$ will be called a *derivator*. We define the pseudometric $d_g(x,y) := |g(x) - g(y)|$ and the *g-ball of center x and radius r* as $B_g(x,r) := \{y \in I \ : \ d_g(x,y) < r\}$. Furthermore, we define the *g-topology* as follows:

$$\tau_g := \left\{U \subset X \ : \ \forall x \in U \, \exists r \in \mathbb{R}^+,\ B_g(x,r) \subset U\right\}.$$

Let

$$C_g := \{t \in \mathbb{R} \ : \ g \text{ is constant on } (t-\varepsilon, t+\varepsilon) \text{ for some } \varepsilon > 0\},$$
$$D_g := \{t \in \mathbb{R} \ : \ \Delta g(t) > 0\},$$

where $\Delta g(t) := g(t^+) - g(t)$, and $g(t^+)$ is the right limit of g at t. C_g is open in the usual topology, and can therefore be expressed as

$$C_g = \bigcup_{n \in \Lambda} (a_n, b_n),$$

Basic definitions

A function $f\ :I\ :\ \mathbb{R}$ is *g-continuous* at a point $t\in I$ if for every $\varepsilon>0$ there exists $\delta>0$ such that

$$|f(t)-f(s)|<\varepsilon,\ \text{for all } t,s\in I,\ |g(t)-g(s)|<\delta.$$

The *g-uniform continuity* is defined analogously to the usual case.
We denote by $\mathcal{C}_g(I;\mathbb{F})$ the set of bounded g-continuous functions on I.
$\mathcal{BC}_g(I,\mathbb{F})$ is a Banach space with the supremum norm. $\mathcal{BUC}_g(I,\mathbb{F})$ (bounded uniformly g-continuous functions) is a Banach subspace of $\mathcal{BC}_g(I,\mathbb{F})$.

Properties inherited by g-continuous functions

Proposition

If $f : [a,b] \to \mathbb{F}$ is g-continuous, then:

- *f is left continuous at each $t \in (a,b]$;*
- *if g is continuous in $t \in [a,b)$, then so is f;*
- *if g is constant at some $[\alpha,\beta]$, then so is f.*

Compactness results

Theorem (Ascoli-Arzelà in $\mathcal{BC}_g([a,b])$)

Let $S \subset \mathcal{BC}_g([a,b],\mathbb{F})$, then S is precompact if and only if

1. $S(t) = \{f(t) \ : \ f \in S\}$ *is bounded for every* $t \in [a,b]$;
2. S *is* g-*equicontinuous;*
3. S *is* g-*stable.*

Theorem (Ascoli-Arzelà in $\mathcal{BUC}_g([a,b])$)

Sea $S \subset \mathcal{BUC}_g([a,b],\mathbb{F})$, then S is precompact if and only if

1. $S(t)$ *is bounded for every* $t \in [a,b]$;
2. S *es uniformly* g-*equicontinuous.*

Integration

We will write as μ_g the Lebesgue-Stieltjes measure associated with g given by

$$\mu_g([c,d)) = g(d) - g(c), \quad c,d \in \mathbb{R},\ c < d.$$

We denote by $\mathcal{L}^1_g(X,\mathbb{F})$ the set of Lebesgue-Stieltjes μ_g-integrable functions on a μ_g-measurable set X with values in $\mathbb{F}$, whose integral we write as

$$\int_X f(s)\,\mathrm{d}\,\mu_g(s), \quad f \in \mathcal{L}^1_g(X,\mathbb{F}).$$

Continuous and jump parts of a derivator

For a derivator g, we define $g^B : \mathbb{R} \to \mathbb{R}$ as

$$g^B(t) := \begin{cases} \displaystyle\sum_{s\in[0,t)\cap D_g} \Delta g(s), & t > 0, \\ \displaystyle -\sum_{s\in[t,0)\cap D_g} \Delta g(s), & t \leqslant 0, \end{cases}$$

which is also a derivator.
So is the function $g^C : \mathbb{R} \to \mathbb{R}$ given by

$$g^C(t) := g(t) - g^B(t).$$

Moreover g^C is continuous. We refer to g^C and g^B as the *continuous* and *jump* parts of g, respectively.

The Stieltjes derivative

Definition

We define *the Stieltjes derivative* of a function $f:\mathbb{R}\to\mathbb{R}$ *with respect to g* at a point $t\in\mathbb{R}$ as

$$f_g'(t)=\begin{cases}\displaystyle\lim_{s\to t}\frac{f(s)-f(t)}{g(s)-g(t)}, & t\notin D_g\cup C_g,\\ \displaystyle\lim_{s\to t^+}\frac{f(s)-f(t)}{g(s)-g(t)}, & t\in D_g,\\ \displaystyle\lim_{s\to b_n^+}\frac{f(s)-f(b_n)}{g(s)-g(b_n)}, & t\in C_g,\ t\in(a_n,b_n),\end{cases}$$

where a_n, b_n are the extremes of each connected component of C_g, provided that the corresponding limits exist. In that case, we say that f es *g-differentiable at t*.

The fundamental theorem of calculus at any point

Why this definition and not another? Because *it is the only definition that makes true the Fundamental Theorem of Calculus* for g-continuous functions.

Theorem (Fundamental theorem of calculus - derivative of the integral at each point)

Let $f \in \mathcal{BC}_g([a,b],\mathbb{F})$. Then f is μ_g-integrable and, if

$$F(t) := \int_{[a,t)} f \,\mathrm{d}\,\mu_g$$

for $t \in [a,b]$, then F is g-differentiable and

$$F_g'(t) = f(t) \text{ for } t \in [a,b).$$

Notation: $$t^* = \begin{cases} t, & t \notin C_g, \\ b_n, & t \in (a_n, b_n) \subset C_g, \end{cases}$$

Proposition (Basic differentiation rules)

Let $t \in [a,b]$. If $f_1, f_2 : [a,b] \to \mathbb{F}$ are g-differentiable at t, the:

- *(Linearity). The function $\lambda_1 f_1 + \lambda_2 f_2$ is g-differentiable at t for every $\lambda_1, \lambda_2 \in \mathbb{F}$ and*

$$\left(\lambda_1 f_1 + \lambda_2 f_2\right)'_g(t) = \lambda_1 \left(f_1\right)'_g(t) + \lambda_2 \left(f_2\right)'_g(t).$$

- *(Product rule). The product $f_1 f_2$ is g-differentiable at t and*

$$\left(f_1 f_2\right)'_g(t) = \left(f_1\right)'_g(t) f_2(t^*) + \left(f_2\right)'_g(t) f_1(t^*) + \left(f_1\right)'_g(t)\left(f_2\right)'_g(t)\Delta g(t^*).$$

g-absolute continuity

A function $F:\mathbb{R}\to\mathbb{R}$ es *g-absolutely continuous*) if for each $\varepsilon>0$ there is $\delta>0$ such that for any family $\left\{\left(a_n,b_n\right)\right\}_{n=1}^{m}$ of open subintervals the inequality

$$\sum_{n=1}^{m}\left(g\left(b_n\right)-g\left(a_n\right)\right)<\delta$$

implies

$$\sum_{n=1}^{m}\left|F\left(b_n\right)-F\left(a_n\right)\right|<\varepsilon.$$

Fundamental theorem of calculus at μ_g-almost every point

Theorem (Fundamental theorem of calculus - derivative of integral)

Suppose that $f:[a,b]\longrightarrow\mathbb{R}$ *is integrable in* $[a,b]$ *with respect to* μ_g *and consider its indefinite Lebesgue-Stieltjes integral*

$$F(x)=\int_{[a,x)} f\,\mathrm{d}\,\mu_g \quad \text{for all } x\in[a,b).$$

Then $F_g'(x)=f(x)$ *for* μ_g*-almost every* $x\in[a,b]$.

Fundamental theorem of calculus at μ_g-almost every point

Theorem (Fundamental theorem of calculus - integral of the derivative)

A function $F:[a,b]\longrightarrow\mathbb{R}$ is g-absolutely continuous at $[a,b]$ if and only if the following three conditions are met:

1. *There exists $F'_g(x)$ for μ_g-almost every $x\in[a,b]$;*
2. $F'_g\in\mathcal{L}^1_g([a,b))$; *and*
3. *for every $x\in[a,b]$ Barrow's rule holds:*

$$F(x)=F(a)+\int_{[a,x)}F'_g(x)\,\mathrm{d}\,\mu_g.$$

Higher order derivation

We can now consider higher order derivatives.

Definition

$$\mathcal{BC}_g^k([a,b];\mathbb{F}) := \{ f \in \mathcal{BC}_g^{n-1}([a,b];\mathbb{F}) : \; f_g^{n)} \in \mathcal{BC}_g([a,b];\mathbb{F}) \}.$$

$$\mathcal{BC}_g^\infty([a,b];\mathbb{F}) := \bigcap_{n\in\mathbb{N}} \mathcal{BC}_g^k([a,b];\mathbb{F}).$$

The differential problem

Now we will consider the problem

$$\begin{cases} u_g'(t) + b(t)u(t) = f(t, u(t)), \quad t \in [0, T], \\ u(0) = u(T) + k\,B(u), \end{cases} \tag{P}$$

where k is a constant and B is a linear functional.

The g-exponential function

Let $h \in L^1_g([a,b),\mathbb{R})$. Let us define the *g-exponential function* $e_h(\cdot;a) : [a,b] \to (0,\infty)$ for every $t \in [a,b]$ as

$$e_h(t;a) = \prod_{s\in[a,t)\cap D_g} \left(1 + h(s)\,\Delta^+ g(s)\right) \exp\left(\int_{[a,t)\setminus D_g} h(s)\,\mathrm{d}\,\mu_g\right).$$

The g-exponential function

Lemma

Let $h \in L^1_g([a,b),\mathbb{R})$. *If* $1 + h(t)\Delta^+ g(t) \neq 0$ *for all* $t \in [a,b) \cap D_g$, *then* $e_h(\cdot;a)$ *never vanishes, if* $1 + h(t)\Delta^+ g(t) > 0$ *for all* $t \in [a,b) \cap D_g$ *then* $e_h(\cdot;a) > 0$.

Theorem

Let $h \in L^1_g([a,b),\mathbb{R})$. *The function*

$$x(t) := x_0 e_h(t;a),$$

is the unique g*–absolutely continuous solution of the initial value problem*

$$x_g'(t) = h(t)x(t) \quad \text{for } |\mu_g|\text{-almost every } t \in [a,b), \quad x(a) = x_0.$$

The Green's Function

Lemma

Assume that $f \in \mathcal{L}^1_g([0,T),\mathbb{R})$, $b \in \mathcal{L}^1_g([0,T),\mathbb{R})$, $1 - b(t)\Delta^+ g(t) \neq 0$ *for all* $t \in [0,T)$, *and* $1 - e_{-b}(T;0) \neq 0$. *Then*

$$v(t) = \int_{[0,T)} h_b(t,s) \frac{f(s)}{1 - b(s)\Delta^+ g(s)} \, \mathrm{d}\,\mu_g(s),$$

$$\text{where} \quad h_b(t,s) = \frac{e_{-b}(t;0)e_{-b}(s;0)^{-1}}{1 - e_{-b}(T;0)} \cdot \begin{cases} 1, & 0 \leqslant s < t \leqslant T, \\ e_{-b}(T;0), & 0 \leqslant t \leqslant s \leqslant T, \end{cases}$$

is the unique solution of the problem

$$\begin{aligned} u_g'(t) + b(t)u(t) &= f(t), \quad \text{for } \mu_g - \text{a.a. } t \in [0,T], \\ u(0) - u(T) &= 0, \end{aligned}$$

The Green's Function

Theorem

Assume that $f \in \mathcal{L}^1_g([0,T),\mathbb{R})$, $b \in \mathcal{L}^1_g([0,T),\mathbb{R})$, $1-b(t)\Delta^+ g(t) \neq 0$ *for all* $t \in [0,T)$, $1-e_{-b}(T;0) \neq 0$, $k \in \mathbb{R}$, *and* $B : \mathcal{BC}_g([0,T],\mathbb{R}) \to \mathbb{R}$ *be a linear bounded operator such that* $1 \neq kB(h_b(\cdot,0))$. *Then*

$$u(t) = \int_{[0,T)} H(t,s,k,b)\frac{f(s)}{1-b(s)\Delta^+ g(s)}\,\mathrm{d}\,\mu_g(s)$$

where $H(t,s,k,b) = h_b(t,s) + \dfrac{k h_b(t,0)}{1-kB(h_b(\cdot,0))}B(h_b(\cdot,s)),\ (t,s) \in [0,T)^2,$

is the unique solution of problem

$$\begin{aligned} u'_g(t) + b(t)u(t) &= f(t), \quad \text{for } \mu_g - \text{a.a. } t \in [0,T], \\ u(0) - u(T) &= kB(u), \end{aligned}$$

The integral operator

We will consider the operator

$$\mathcal{T}x(t)=\int_{[0,T)}H(t,s,k,b)\frac{f(s,x(s))}{1-b(s)\Delta^+g(s)}\,\mathrm{d}\,\mu_g(s).$$

Let $g:\mathbb{R}\to\mathbb{R}$ be a nondecreasing and left-continuous function, $J\subset\mathbb{R}$ be an interval and $X\subset\mathbb{R}^m$. A function $f:J\times X\to\mathbb{R}^n$ is said to be *g-Carathéodory* if the following properties are satisfied:

1. $f(\cdot,x)$ is μ_g-measurable for all $x\in X$,
2. $f(t,\cdot)$ is continuous for μ_g – a.a. $t\in J$,
3. for all $r>0$ there exists $p_r\in\mathcal{L}^1_g(J,[0,\infty))$ such that

$$\|f(t,x)\|\leqslant p_r(t),\ \mu_g-\text{a.a. } t\in J,\ x\in X,\ \|x\|\leqslant r.$$

The integral operator

Lemma

Let X be a nonempty subset of $\mathbb{R}^n$ and $f : [0,T]\times X \to \mathbb{R}^n$ a g-Carathéodory function. Then, for every $x \in \mathcal{BC}_g([0,T],X)$, the map $f(\cdot, x(\cdot))$ is in $\mathcal{L}^1_g([0,T), \mathbb{R}^n)$.

Theorem

Let $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ be a g-Carathéodory function. Then $\mathcal{T}$ maps $\mathcal{BC}_g([0,T),\mathbb{R})$ into $\mathcal{BC}_g([0,T],\mathbb{R})$ and is continuous and compact.

Fixed point index results

We will be assuming the following:

1. $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is g-Carathéodory function such that

$$f(t,x) \geqslant 0 \quad \text{for } t \in [0,T],\ x \geqslant 0.$$

2. $b \in \mathcal{L}^1_g([0,T),\mathbb{R})$ is such that

$$b(t) > 0 \quad \mu_g\text{-a.a. } t \in [0,T],$$

   and

$$1 - b(t)\Delta^+ g(t) > 0 \quad \forall t \in [0,T].$$

3. $B : \mathcal{BC}_g([0,T],\mathbb{R}) \to \mathbb{R}$ is bounded linear functional such that

$$u(t) \geqslant 0, \forall t \in [0,T] \Rightarrow B(u) \geqslant 0$$

   and we assume $k \geqslant 0$ and $1 - kB(h_b(\cdot,0)) > 0$.

Green's functions and cones

Theorem

The Green's function H is positive on $[0,T]\times[0,T]$.

Now we consider the cone

$$P := \{x \in \mathcal{BC}_g\left(\left[0,T\right],\mathbb{R}\right) \,:\, x(t) \geqslant e_{-b}(T;0)\,\|x\|_\infty\,,\; t\in[0,T]\}.$$

Theorem

$\mathcal{T}(P)\subset P$.

First, for some $\rho > 0$, we define the sets

$$M^\rho := \left\{ u \in P : \inf_{t\in[0,T)} u(t) < \rho \right\},$$

$$N^\rho := \left\{ u \in P : \sup_{t\in[0,T)} u(t) < \rho \right\}.$$

We also define a new function $n : \mathbb{R} \to \mathbb{R}$ by

$$n(\rho) := \frac{\rho}{e_{-b}(T;0)}, \quad \rho > 0.$$

Lemma

For every $\rho > 0$ we have $N^\rho \subset M^\rho \subset N^{n(\rho)}$.

Fixed point index results

Lemma

Assume that there exists $\rho > 0$ such that

$$f^{\rho}\int_{[0,T)}\frac{1}{1-e_{-b}(T;0)}\left(1+\frac{k}{1-kB\left(h_b\left(\cdot,0\right)\right)}B\left(h_b\left(\cdot,s\right)\right)\right)\mathrm{d}\,\mu_g\left(s\right)<1,$$

where

$$f^{\rho}=\frac{1}{\rho}\sup\left\{\frac{f\left(t,u(t)\right)}{1-b(t)\Delta^{+}g(t)};t\in[0,T),u\in P,\inf_{t\in[0,T)}u(t)=\rho\right\}.$$

Then $i_P\left(\mathcal{T},M^{\rho}\right)=1$.

Fixed point index results

Lemma

Assume that there exists $\rho > 0$ such that

$$f_\rho \int_{[0,T)} \frac{e_{-b}(T;0)}{1-e_{-b}(T;0)} \left(1 + \frac{k}{1-kB\left(h_b\left(\cdot,0\right)\right)} B\left(h_b\left(\cdot,s\right)\right)\right) \mathrm{d}\,\mu_g\left(s\right) > 1,$$

where

$$f_\rho = \frac{1}{\rho} \inf\left\{ \frac{f\left(t,u(t)\right)}{1-b(t)\Delta^+ g(t)}; t \in [0,T), u \in P, \sup_{t\in[0,T)} u(t) = \rho \right\}.$$

Then $i_P\left(\mathcal{T}, N^\rho\right) = 0$.

Theorem

Problem (P) has at least one nontrivial solution in P if one of the following conditions hold:

(S_1) *there exist $\rho_1, \rho_2 > 0$ with $\rho_2 > \rho_1$ such that*

$$f_{\rho_1} \cdot \int_{[0,T)} \frac{e_{-b}(T;0)}{1-e_{-b}(T;0)} \left(1 + \frac{k}{1-kB\left(h_b(\cdot,0)\right)} B\left(h_b(\cdot,s)\right)\right) \mathrm{d}\,\mu_g(s) > 1,$$

$$f^{\rho_2} \cdot \int_{[0,T)} \frac{1}{1-e_{-b}(T;0)} \left(1 + \frac{k}{1-kB\left(h_b(\cdot,0)\right)} B\left(h_b(\cdot,s)\right)\right) \mathrm{d}\,\mu_g(s) < 1.$$

(S_2) *there exist $\rho_1, \rho_2 > 0$ with $\rho_2 > n(\rho_1)$ such that*

$$f^{\rho_1} \cdot \int_{[0,T)} \frac{1}{1-e_{-b}(T;0)} \left(1 + \frac{k}{1-kB\left(h_b(\cdot,0)\right)} B\left(h_b(\cdot,s)\right)\right) \mathrm{d}\,\mu_g(s) < 1,$$

$$f_{\rho_2} \cdot \int_{[0,T)} \frac{e_{-b}(T;0)}{1-e_{-b}(T;0)} \left(1 + \frac{k}{1-kB\left(h_b(\cdot,0)\right)} B\left(h_b(\cdot,s)\right)\right) \mathrm{d}\,\mu_g(s) > 1.$$

Example

Consider the following problem

$$\begin{cases} u_g'(t) + bu(t) = |u(t)|^\alpha, & t \in [0,2], \\ u(0) = u(2) + k \int_{[0,2)} u(t)\, \mathrm{d}\mu_g(t), & \end{cases} \tag{(S)}$$

where $b \in (0,1)$, $\alpha > 1$,

$$g(t) = \begin{cases} t, & t \in [0,1], \\ t+1, & t \in (1,2] \end{cases}$$

and $k \in [0, k_0)$, where

$$k_0 = \frac{1-(1-b)\mathrm{e}^{-2b}}{\int_{[0,2)} \hat{e}_{-b}(s)\, \mathrm{d}\mu_g(s)}, \quad \hat{e}_{-b}(t) = \begin{cases} \mathrm{e}^{-bt}, & t \in [0,1], \\ (1-b)\mathrm{e}^{-bt}, & t \in (1,2]. \end{cases}$$

It is clear that g is a nondecreasing and left continuous function such that

$$D_g = \{1\} \quad \text{and} \quad \Delta^+ g(1) = 1.$$

We take

$$0 < \rho_1 < \left(\frac{(1-b)^{\alpha+1} \, e^{-2\alpha b}}{I} \right)^{\frac{1}{\alpha-1}},$$

$$\rho_2 > \left(\frac{1}{(1-b)^{\alpha+1} \, e^{-2b(\alpha+1)} \cdot I} \right)^{\frac{1}{\alpha-1}}.$$

where $I =$

$$\int_{[0,2)} \frac{1}{1-(1-b)\,e^{-2b}} \left(1 + \frac{k}{1 - k \int_{[0,2)} h_b(t,0) \, d\mu_g(t)} \int_{[0,2)} h_b(t,s) \, d\mu_g(t) \right) d\mu_g(s).$$

The assumptions in (S_2) hold and problem (S) has at least one nontrivial solution in P.

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