Invariant graphs and dynamics of strongly dissipative planar piecewise linear continuous maps

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Invariant sets play a fundamental role in the study of dynamical systems. Their importance is evident for systems possessing a first integral (an invariant), but they are also essential for strongly dissipative systems, which are the main subject of this talk. We consider *strongly dissipative maps* given by *continuous piecewise-linear planar maps*. Under suitable conditions, we will show that these maps possess *invariant compact graphs* that capture the global dynamics. Recall that a graph is a one-dimensional connected piecewise-linear set consisting of a finite union of segments, lines, and rays, called edges.

As we will see, the dynamics on these invariant graphs may be chaotic. The existence of such invariant graphs has recently been investigated in several settings. Numerical evidence [6] and also rigorous existence results [5] have been obtained in the context of border-collision bifurcations; and invariant graphs with a wide variety of topological types have been described for specific families of maps [2, 4].

In the first part of the talk, I will present some results obtained together with A. Cima, A. Gasull and F. Mañosas in those last references, for the very particular family of maps

$$F(x,y) = (|x| - y - 1, x - |y| + b)\,,$$

where $b \in \mathbb{R}$. When $b < 0$, we will see that these maps possess one-dimensional invariant graphs, whose topology depends on the parameter b, and that these graphs capture the global dynamics of the system. For certain parameter values, chaotic dynamics appear on these graphs, giving rise to an intermediate dynamical regime between regular behavior and full-plane chaos. To study this phenomenon, we analyze in detail the *topological entropy* of the map restricted to the invariant graph as a function of the bifurcation parameter. Topological entropy is a measure of the complexity of a dynamical system. When the topological entropy is positive, the dynamics is highly complex and, in particular, chaotic in the sense of Li and Yorke; see [1] for a general reference on the topic.

Motivated by the limitations of numerical simulations in detecting the complex dynamics on these graphs, I will also present a result showing that, for some parameter values, there exists a full-measure subset of the invariant graph on which all orbits converge to at most three ω-limit sets. When the parameter is rational, these ω-limit sets are periodic orbits. Therefore, numerical simulations do not reveal the chaotic dynamics, although it is actually present.

The first part of the talk will be mainly expository. I will briefly recall the notion of topological entropy and explain how it can be computed for maps restricted to invariant graphs.

In the second part of the talk, I will show that the existence of invariant graphs is a rather general phenomenon by studying a simple class of two-zone continuous piecewise-linear planar maps:

$$F(x,y) = \begin{cases} F_+(x,y) & \text{if } x > 0 \\ F_-(x,y) & \text{if } x \le 0 \end{cases} \tag{1}$$

with $F_\pm(x,y) = A_\pm \cdot (x,y)^T + b$, where $A_\pm$ are matrices in $\mathbb{R}^{2\times 2}$ satisfying $A^+(0,1) = A^-(0,1)$ and $b \in \mathbb{R}^2$.

I will present some results from our recent work [3], carried out in collaboration with A. Cima, A. Gasull and F. Mañosas. Our main result states that, when the fixed point of F_+ is either a virtual focus, a real repelling focus, or a virtual center, there exists a finite invariant graph Ω (not necessarily compact) that captures the global dynamics, and one of the following alternatives holds: either (i) there exists a compact invariant subgraph $\Gamma \subset \Omega$ such that every bounded orbit of $F|\Omega$ is contained in Γ; or (ii) there exists a compact, non-invariant subgraph $\Gamma \subset \Omega$ such that every orbit starting outside Γ escapes to infinity, while every orbit that does not escape is contained in a compact invariant subset $\Delta \subset \Gamma$.

The analysis of the dynamics of these kind of maps reveals the coexistence of regular and chaotic behavior, together with a rich and complex bifurcation structure. As an illustration, I will present a simple example for which F_+ has a real focus and the dynamics of F on the invariant graph has positive topological entropy and a non-degenerate rotation interval.

References

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