Asymptotic decomposition of stochastic semigroups and its applications

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Many natural phenomena are governed by conservation laws of some quantities, e.g., mass, energy, volume, total electric charge, number of molecules in a diffusion process, length of polymers in polymerization and fragmentation processes. When studying such objects we are interested in how the distributions of a given quantity change. The space X, in which we consider given phenomena, is usually equipped with some standard measure m, e.g. if X is a countable space, then $m(A)$ is the number of elements of A; if X is a subset of $\mathbf{R}^d$, then m is the Lebesgue measure. Typically, measures that are absolutely continuous with respect to m (i.e., measures that have density with respect to m) transform into absolutely continuous measures. Therefore, it is convenient to restrict the consideration to the evolution of densities, which is well described by stochastic operators and semigroups. Formally, a *stochastic operator* or *Markov operator* P is a linear operator defined on some L^1 space which has the property: if $f \in D$, then $Pf \in D$, where D is the set of all densities, i.e. non-negative functions with integral one. A *stochastic semigroup* $\{P(t)\}_{t\geqslant 0}$ is a C_0-semigroup of stochastic operators, i.e. $P(t)$ is a stochastic operator for $t \geqslant 0$, $P(0) = \mathrm{Id}$, $P(t+s) = P(t)P(s)$ and $\lim_{t\downarrow 0} P(t)f = f$ for $f \in L^1$.

Stochastic semigroups have been intensively studied because they play a special role in applications [1]. They are used to investigate the long-time behaviour of the distributions of Markov chains, diffusion processes and piecewise deterministic processes. They can be also used to study asymptotic stability of partial differential equations with non-local terms.

We will present some general results concerning the long-time behaviour of stochastic semigroups [2,3,4]. In particular, we will state a theorem on the decomposition of a stochastic semigroup into asymptotically stable and sweeping components. A stochastic semigroup $\{P(t)\}_{t\geqslant 0}$ is called *asymptotically stable* if there is an invariant density f_* such that $\lim_{t\to\infty} \|P(t)f - f_*\| = 0$ for every $f \in D$. A stochastic semigroup $\{P(t)\}_{t\geqslant 0}$ is called *sweeping* with respect to a set $B \in \Sigma$ if $\lim_{t\to\infty} \int_B P(t)f(x)\, m(dx) = 0$ for every $f \in D$. It is clear that sweeping is an opposite property to asymptotic stability.

We apply these results to study asymptotic stability of semigroups generated by piecewise deterministic Markov processes (PDMPs). A PDMP is a continuous-time Markov process $\xi(t)$ with values in some metric space X and there is an increasing sequence of random times (t_n), called jump times, such that sample paths (trajectories) of $\xi(t)$ are defined in a deterministic way in each interval (t_n, t_{n+1}). PDMPs is a large family of different stochastic processes which includes continuous-time Markov chains, deterministic processes with jumps, dynamical systems with random switching and stochastic billiards.

We illustrate mathematical results by applications to a gene expression model [5], an immune system dynamics [6], and stochastic billiards [7].

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